AP 2024–01

Chair of Applied Stochastics and Risk Management

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Working Paper

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A differentiation lemma for càdlàg and càglàd functions

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Abstract

In this paper we generalize the differentiation lemma of measure and integration theory for those functions that are not differentiable on both sides in finitely many points, so that the interchange of the one-sided differential operator and the integration with respect to an arbitrary measure is guaranteed. The central prerequisite is that these are càdlàg or càglàd functions.

As can be seen in the literature (e.g. Bauer (1992), Elstrodt (2018), Klenke (2020)), the differentiation lemma is a relevant theorem in measure and integration theory. It is a tool for mathematical proof. However, it does not apply to functions that are not differentiable in finitely many points in the first argument. In this paper, the differentiation lemma is generalized for such functions as long as one-sided differentiability is guaranteed. Therefore, we consider the one-sided differential operators and extend the preconditions accordingly. In particular, we require that the functions are càdlàg or càglàd functions as long as the second argument is kept constant.

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1. Definitions

First, we introduce some definitions.

Definition 1. Let $I \subseteq \mathbb{R}$ be a nondegenerate intervall. We denote the set of all càdlàg respectively càglàd functions $f : I \to \mathbb{R}$ by $\mathcal{C}^+(I)$ respectively $\mathcal{C}^-(I)$.

Definition 2. Let $D \subseteq \mathbb{R}$ and $(\Omega, \mathcal{A}, \mu)$ be a measure space. We denote the set of all μ-integrable functions $f : \Omega \to D$ by $\mathcal{L}^1_\mu(D)$.

Definition 3. We denote the right-sided respectively left-sided differential operator with respect to x by ∂_x^+ respectively ∂_x^-.

2. A generalization of the differentiation lemma

The following theorem is a generalization of the differentiation lemma. It assumes four properties. For $n = 0$ in the third property, we obtain the classical case with a weaker requirement for the derivative.

Theorem 4. *Let $I \subseteq \mathbb{R}$ be a nondegenerate interval and $f : I \times \Omega \to \mathbb{R}$ be a function with the following properties:*

1. $(f(\cdot,\omega) \in \mathcal{C}^+(I)\ \forall\, \omega \in \Omega) \vee (f(\cdot,\omega) \in \mathcal{C}^-(I)\ \forall\, \omega \in \Omega)$.
2. $\Big(f(x+,\cdot) \in \mathcal{L}^1_\mu(\mathbb{R})\ \forall\, x \in I \backslash \sup I\Big) \wedge \Big(f(x-,\cdot) \in \mathcal{L}^1_\mu(\mathbb{R})\ \forall\, x \in I \backslash \inf I\Big)$.
3. $\exists\, n \in \mathbb{N}_0, (x_i)_{i\in\{1,\dots,n\}} \subset I\ \forall\, \omega \in \Omega :$
$$\left|\frac{\partial}{\partial x} f(x,\omega)\right|, \left|\partial_x^\pm f(x_j,\omega)\right| < \infty\ \forall\, x \in I \backslash (x_i)_{i\in\{1,\dots,n\}}, j \in \{1,\dots,n\}.$$
4. $\forall\, a, b \in I, a < b\ \exists\, h \in \mathcal{L}^1_\mu(\mathbb{R}_0^+)\ \forall\, \omega \in \Omega :$
$$\left(\left|\partial_x^+ f(x,\omega)\right| \le h(\omega)\ \forall\, x \in [a,b)\right) \wedge \left(\left|\partial_x^- f(x,\omega)\right| \le h(\omega)\ \forall\, x \in (a,b]\right).$$

Then

$$\partial_x \int_\Omega f(x,\omega)\, d\mu(\omega) = \int_\Omega \partial_x f(x,\omega)\, d\mu(\omega)\ \forall\, x \in I \backslash \{\alpha\}$$

with

$$(\partial_x, \alpha) := \begin{cases} (\partial_x^+, \sup I) & \text{if } f(\cdot,\omega) \in \mathcal{C}^+(I)\ \forall\, \omega \in \Omega \\ (\partial_x^-, \inf I) & \text{if } f(\cdot,\omega) \in \mathcal{C}^-(I)\ \forall\, \omega \in \Omega \end{cases}.$$

Proof. First let $n = 0$. We obtain

$$\frac{\partial}{\partial x}\int_{\Omega} f_{|[a,b]\times\Omega}(x,\omega)\, d\mu(\omega) = \int_{\Omega} \frac{\partial}{\partial x} f_{|[a,b]\times\Omega}(x,\omega)\, d\mu(\omega)\ \forall\, x \in [a,b]$$

for each $a, b \in I$ with $a < b$, which corresponds to the statement. Now let $n > 0$. Assume, w.l.o.g.,

$$x_i < x_{i+1}\ \forall\, i \in \{1, ..., n-1\}.$$

We define $(I_i)_{i\in\{0,...,n\}}$ by the following case distinction:

1. For the case $f(\cdot,\omega) \in \mathcal{C}^+(I)\ \forall\, \omega \in \Omega$, we define

$$\begin{aligned} I_0 &:= \{x \in I \mid x < x_1\}, \\ I_i &:= [x_i, x_{i+1})\ \forall\, i \in \{1, ..., n-1\} \text{ and} \\ I_n &:= \{x \in I \mid x \geq x_n\}. \end{aligned}$$

2. For the case $f(\cdot,\omega) \in \mathcal{C}^-(I)\ \forall\, \omega \in \Omega$, we define

$$\begin{aligned} I_0 &:= \{x \in I \mid x \leq x_1\}, \\ I_i &:= (x_i, x_{i+1}]\ \forall\, i \in \{1, ..., n-1\} \text{ and} \\ I_n &:= \{x \in I \mid x > x_n\}. \end{aligned}$$

With

$$\left|\partial_x^- f(x_1,\omega)\right| < \infty\ \forall\, \omega \in \Omega \Rightarrow x_1 \neq \inf I$$

and

$$\left|\partial_x^+ f(x_n,\omega)\right| < \infty\ \forall\, \omega \in \Omega \Rightarrow x_n \neq \inf I,$$

we define $f_0, f_n : I \times \Omega \to \mathbb{R}$ by

$$f_0(x,\cdot) := \begin{cases} f(x,\cdot) & \text{if } x < x_1 \\ f(x_1-,\cdot) + (x - x_1)\partial_x^- f(x_1,\cdot) & \text{if } x \geq x_1 \end{cases} \text{ and}$$

$$f_n(x,\cdot) := \begin{cases} f(x_n+,\cdot) + (x - x_n)\partial_x^+ f(x_n,\cdot) & \text{if } x \leq x_n \\ f(x,\cdot) & \text{if } x > x_n \end{cases}.$$

We further define $(f_i : I \times \Omega \to \mathbb{R})_{i\in\{1,...,n-1\}}$ by

$$f_i(x,\cdot) := \begin{cases} f(x_i+,\cdot) + (x-x_i)\partial_x^+ f(x_i,\cdot) & \text{if } x \le x_i \\ f(x,\cdot) & \text{if } x \in (x_i, x_{i+1}) \\ f(x_{i+1}-,\cdot) + (x-x_{i+1})\partial_x^- f(x_{i+1},\cdot) & \text{if } x \ge x_{i+1} \end{cases}.$$

This gives us the following five properties for f_i and I_i with $i \in \{0,...,n\}$:

1. $f_i(x,\cdot) \in \mathcal{L}^1_\mu(\mathbb{R}) \;\forall\, x \in I$
2. $\left|\frac{\partial}{\partial x} f_i(x,\omega)\right| < \infty \;\forall\, x \in I, \omega \in \Omega$
3. $\left|\frac{\partial}{\partial x} f_i(x,\omega)\right| \le h(\omega) \;\forall\, x \in I, \omega \in \Omega$
4. $f_i(x,\cdot) = f(x,\cdot) \;\forall\, x \in I_i$
5. $\partial_x \mathbb{1}_{x\in I_i} f_i(x,\omega) = \mathbb{1}_{x\in I_i} \frac{\partial}{\partial x} f_i(x,\omega) \;\forall\, x \in I\backslash\{\alpha\}, \omega \in \Omega$

Together with the (differentiation lemma), we obtain

$$\begin{aligned} \int_\Omega \partial_x f(x,\omega)\, d\mu(\omega) &= \int_\Omega \partial_x \sum_{i=0}^n \mathbb{1}_{x\in I_i} f_i(x,\omega)\, d\mu(\omega) \\ &= \sum_{i=0}^n \mathbb{1}_{x\in I_i} \int_\Omega \frac{\partial}{\partial x} f_i(x,\omega)\, d\mu(\omega) \\ &= \sum_{i=0}^n \mathbb{1}_{x\in I_i} \frac{\partial}{\partial x} \int_\Omega f_i(x,\omega)\, d\mu(\omega) \\ &= \partial_x \int_\Omega f(x,\omega)\, d\mu(\omega) \end{aligned}$$

for each $x \in I\backslash\{\alpha\}$. □

A. Appendix

Theorem 5 (differentiation lemma Bauer (1992)). *Let $I \subseteq \mathbb{R}$ be a nondegenerate interval and $f : I \times \Omega \to \mathbb{R}$ be a function with the following properties:*

1. $f(x,\cdot) \in \mathcal{L}^1_\mu(\mathbb{R}) \;\forall\, x \in I.$
2. $\left|\frac{\partial}{\partial x} f(x,\omega)\right| < \infty \;\forall\, x \in I, \omega \in \Omega.$
3. $\exists\, h \in \mathcal{L}^1_\mu(\mathbb{R}^+_0) : \left|\frac{\partial}{\partial x} f(x,\omega)\right| \le h(\omega) \;\forall\, x \in I, \omega \in \Omega.$

Then

$$\frac{\partial}{\partial x} \int_\Omega f(x,\omega)\, d\mu(\omega) = \int_\Omega \frac{\partial}{\partial x} f(x,\omega)\, d\mu(\omega) \;\forall\, x \in I.$$

References

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Elstrodt, J. (2018): *Mass- und Integrationstheorie*. Berlin, Springer Spektrum: 162.

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