AP 2018–01

Chair of Applied Stochastics and Risk Management

HELMUT SCHMIDT UNIVERSITÄT
Universität der Bundeswehr Hamburg
Faculty of Economic and Social Sciences
Department of Mathematics and Statistics

Working Paper

A Solution to Ellsberg's Paradox

Gabriel Frahm

January 21, 2018

A Solution to Ellsberg's Paradox

Gabriel Frahm

Helmut Schmidt University
Faculty of Economic and Social Sciences
Department of Mathematics and Statistics
Chair of Applied Stochastics and Risk Management
Holstenhofweg 85, D-22043 Hamburg, Germany

URL: www.hsu-hh.de/stochastik
Phone: +49 (0)40 6541-2791
E-mail: frahm@hsu-hh.de

Working Paper
Please use only the latest version of the manuscript. Distribution is unlimited.

Supervised by: Prof. Dr. Gabriel Frahm
Chair of Applied Stochastics and Risk Management

URL: www.hsu-hh.de/stochastik

A Solution to Ellsberg's Paradox*

Gabriel Frahm†
Helmut Schmidt University
Department of Mathematics/Statistics
Chair of Applied Stochastics and
Risk Management

January 21, 2018

Abstract

Ellsberg's famous thought experiments demonstrate that most people prefer less ambiguous alternatives to more ambiguous ones. This apparently violates Savage's Sure-thing Principle. I provide a solution to Ellsberg's paradox. More precisely, I demonstrate that ambiguity aversion can be readily explained by subjectivistic decision theory. The given solution is simple and fits perfectly into Savage's subjectivistic framework. Since ambiguity aversion translates into the subjective probabilities of the decision-maker, they could even be used in order to quantify his ambiguity aversion.

Keywords: Ambiguity, Ellsberg's paradox, Risk, Sure-thing Principle, Uncertainty.

JEL Subject Classification: D81, D91.

---

*I would like to thank very much Philipp Adämmer for our vivid discussions on that topic.
†Phone: +49 40 6541-2791, e-mail: frahm@hsu-hh.de.

1. Motivation

By his famous thought experiments, Ellsberg (1961) argues that most decision-makers are ambiguity averse and concludes that ambiguity aversion contradicts Savage's axioms of subjectivistic decision theory (Savage, 1972). This note provides a solution to Ellsberg's paradox. I do not question ambiguity aversion nor do I develop a new alternative to subjectivistic decision theory. I just demonstrate that ambiguity aversion can be readily explained by Savage's subjectivistic principle of rational choice. The given solution is astonishingly simple and reflects the basic idea of subjective probabilities. Simply put, the Ellsberg paradox disappears as soon as subjective probabilities are no longer treated like objective ones. I hope that the solution provides some fresh insights into the subjectivistic framework.

2. Ellsberg's Thought Experiments

The two-colors experiment is discussed on p. 650 to p. 653, whereas the three-colors experiment is investigated on p. 653 to p. 655 in Ellsberg (1961). I recapitulate Ellsberg's thought experiments not only for convenience, but also to provide a better illustration of the solution to the paradox.

2.1. The Two-Colors Experiment

Consider two urns, each one containing 100 red and black balls. The number of red balls and the number of black balls in Urn I are unknown to the decision-maker,[1] whereas he knows that Urn II contains 50 red and 50 black balls. He can place one of the following bets:

- Red_{I}: Red in Urn I
- Black_{I}: Black in Urn I
- Red_{II}: Red in Urn II
- Black_{II}: Black in Urn II

Then a ball is drawn at random from each urn.[2] In case the decision-maker proves correct, he wins \$100 and otherwise he comes away empty-handed. There are no other costs or benefits.

Ellsberg's decision matrix of the two-colors experiment is given by Table 1, where the first row of colors represents Urn I and the second Urn II.[3] He conjectures that most people prefer Red_{II} to Red_{I} and Black_{II} to Black_{I}, since the number of red (or black) balls in Urn I is ambiguous. Those decision-makers apparently violate Savage's Sure-thing Principle (Savage, 1972, p. 23). Red_{I} agrees with Black_{II} and Black_{I} agrees with Red_{II} on $\{\omega_2, \omega_3\}$, whereas Red_{I} agrees with Red_{II} and Black_{I} agrees with Black_{II} on $\{\omega_1, \omega_4\}$. The Sure-thing Principle states that a rational

[1] There even can be 100 red balls or 100 black balls in Urn I.

[2] According to Ellsberg (1961, p. 650), the decision-maker himself draws the ball from the respective urn. In order to construct the state space of the experiment, we have to take both urns into consideration.

[3] Ellsberg's arguments on the two-colors experiment are only verbal. That is, he uses no decision matrix at all. The given matrix can be considered a canonical state-space representation based on his arguments.

	ω_1	ω_2	ω_3	ω_4
	•	•	•	•
Act	•	•	•	•
Red_{I}	100	100	0	0
Black_{I}	0	0	100	100
Red_{II}	100	0	100	0
Black_{II}	0	100	0	100

Table 1: Ellsberg's decision matrix of the two-colors experiment.

decision-maker who prefers Red_{II} to Red_{I} should also prefer Black_{I} to Black_{II}.[4] Ellsberg holds that ambiguity cannot be expressed by Savage's subjectivistic decision theory.

2.2. The Three-Colors Experiment

Now, there is only one urn, which contains 90 balls. More precisely, it contains 30 red balls and 60 black and yellow balls, but the number of black and the number of yellow balls are unknown to the decision-maker. He can make the following bets:

- I: Red
- II: Black
- III: Red or Yellow
- IV: Black or Yellow

Ellsberg's decision matrix of the three-colors experiment is given by Table 2 (Ellsberg, 1961, p. 654). It turns out to be simpler than the decision matrix of the two-colors experiment. This is because there is only one urn, whereas the two-colors experiment involves two urns.

Ellsberg argues that most people prefer I to II and IV to III. Once again, the reason is ambiguity: The decision-maker knows that the urn contains 30 red balls but he does not know the number of black balls. Thus he prefers I to II. Moreover, he knows that the total number of black and yellow balls is 60 but he does not know the total number of red and yellow balls. For this reason, he prefers IV to III. This is a seeming contradiction to the Sure-thing Principle: I agrees with III and II agrees with IV on $\{\omega_1, \omega_2\}$. Further, I agrees with II and III agrees with IV on $\{\omega_3\}$. Hence, if a rational decision-maker prefers I to II he should also prefer III to IV.

It seems that the decision-makers cannot be rational in the sense of Savage (1972), and Ellsberg (1961, p. 655) claims that, at least for them, it is even impossible to infer qualitative probabilities. He comes to the conclusion that their behavior cannot be explained by expected-utility theory.

[4] Moreover, Ellsberg (1961, p. 651) assumes that most people are indifferent between Red_{I} and Black_{I} as well as between Red_{II} and Black_{II}. In this case, Savage's axioms imply that those people cannot prefer any bet at all.

	Ω		
	ω_1	ω_2	ω_3
Act	●	●	●
I	100	0	0
II	0	100	0
III	100	0	100
IV	0	100	100

Table 2: Ellsberg's decision matrix of the three-colors experiment.

3. The Solution

The solution to Ellsberg's paradox is based on the simple observation that the state spaces of the decision problems that are described in Section 2 are not properly specified. The decision matrices in fact contain (much) more columns, but they have been neglected. Put another way, the corresponding subjective probabilities have been implicitly assumed to be zero.

3.1. Decision Trees

This can be best illustrated by using decision trees. Let us start with the two-colors experiment. Suppose, without loss of generality, that the subjective probabilities are given by the decision tree on the left of Figure 1. The probabilities for Red and for Black may depend on the chosen act: The decision-maker believes that Red occurs with probability 30% if he chooses Red_{I}, but his probability of Red changes to 70% if he chooses Black_{I}. By contrast, his probability of Red is 50% whenever he chooses Red_{II} or Black_{II}.[5] In his thought experiment, Ellsberg (1961, p. 651) precisely describes the situation depicted by the decision tree, i.e., we have that

- $\text{Red}_{\text{I}} \sim \text{Black}_{\text{I}}$ and $\text{Red}_{\text{II}} \sim \text{Black}_{\text{II}}$ but
- $\text{Red}_{\text{II}} \succ \text{Red}_{\text{I}}$ and $\text{Black}_{\text{II}} \succ \text{Black}_{\text{I}}$.

We conclude that the decision-maker, in fact, *is* ambiguity averse. Nonetheless, this phenomenon can be simply explained by means of decision theory—but only if we do not treat subjective probabilities like objective ones.

In general, the probabilities in a decision tree may depend on the actions of the decision-maker. However, it might seem obscure to the reader that we allow the probability of Red to depend on whether the decision-maker chooses Red_{I} or Black_{I}. One could think that the decision-maker ponders on the possible number of red (or black) balls in Urn I before making his decision. After a while he comes to the conclusion that Urn I contains 30 red and 70 black balls. Of course, if this is actually the way he creates his subjective probabilities, it cannot happen that the probability

[5] Actually, even for Urn II the *subjective* probability of Red need not necessarily be $\frac{1}{2}$. For example, the decision-maker might prefer Red to Black (in Urn II) although he knows that the numbers of red and of black balls are equal.

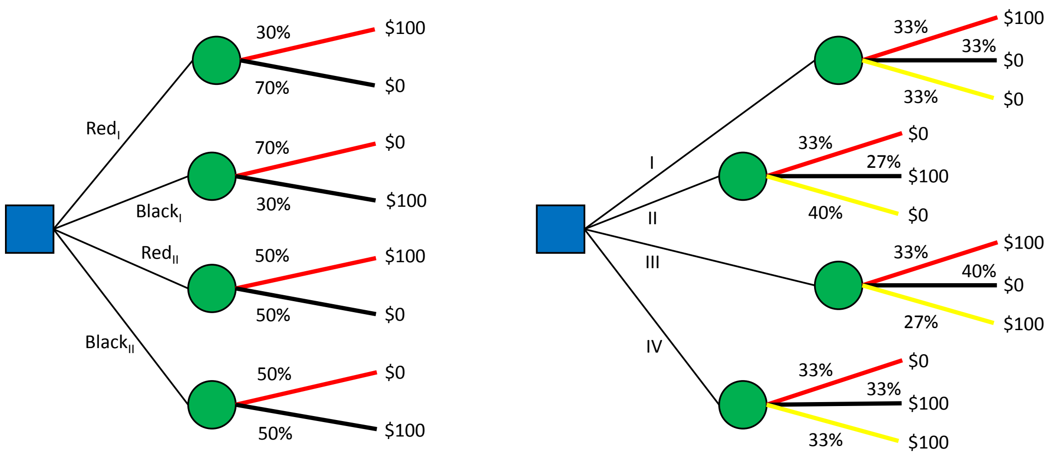

Figure 1: Decision trees of the thought experiments.

of Red depends on whether he chooses Red_{I} or Black_{I}.[6] In this case, he should place his bet on Black rather than Red in Urn I.

This way of thinking is purely frequentistic and has nothing in common with the subjectivistic approach to rational choice. Subjective probabilities are a result of the given preferences of the decision-maker but his individual preferences are not a result of any given probabilities. Since subjectivistic decision theory does not scrutinize the reasoning of the decision-maker, it may well happen that his probabilities change with each single act.[7] Subjective probabilities are not bound to physical laws; they reflect the decision-maker's individual preferences among acts.

The probabilities depicted in the decision tree on the left of Figure 1 describe the whole idea of ambiguity aversion, namely that the decision-maker is pessimistic regarding Urn I. Irrespective of wether he chooses Red or Black—as long as he is betting on Urn I, he believes that the probability of winning is only 30%. This is because he does not know how much red and black balls are in Urn I, and thus he feels uncomfortable when placing a bet on this urn. By contrast, he knows the physical distribution of red and black balls in Urn II, and so his probability of winning amounts to 50% if he is betting on Urn II. Hence, ambiguity aversion translates into subjective probabilities, which means that they could even be used in order to quantify the decision-maker's ambiguity aversion—provided he obeys Savage's axioms of rational choice.

Finally, we can apply the same arguments to the three-colors experiment and obtain, for example, the subjective probabilities in the decision tree on the right of Figure 1. Now, we have that $\text{I} \succ \text{II}$ but $\text{III} \prec \text{IV}$, in accordance with Ellsberg's (1961) hypothesis on p. 654.

3.2. Decision Matrices

Now, I will show that the Sure-thing Principle is neither violated in the two-colors nor in the three-colors experiment by referring to decision matrices. For this purpose, we have to count the number of states in each decision problem. From elementary decision theory we know that, in order to count the states, we must take all branches in the decision tree into account.

[6]The number of red and the number of black balls are fixed, i.e., they cannot be influenced by the decision-maker.
[7]This is not to say that the probability *measure* of the decision-maker, i.e., his prior, depends on his decision.

	Ω															
	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9	ω_{10}	ω_{11}	ω_{12}	ω_{13}	ω_{14}	ω_{15}	ω_{16}
	$\color{red}\bullet$	$\color{red}\bullet$	$\color{red}\bullet$	$\color{red}\bullet$	$\color{red}\bullet$	$\color{red}\bullet$	$\color{red}\bullet$	$\color{red}\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
	$\color{red}\bullet$	$\color{red}\bullet$	$\color{red}\bullet$	$\color{red}\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\color{red}\bullet$	$\color{red}\bullet$	$\color{red}\bullet$	$\color{red}\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
	$\color{red}\bullet$	$\color{red}\bullet$	$\bullet$	$\bullet$	$\color{red}\bullet$	$\color{red}\bullet$	$\bullet$	$\bullet$	$\color{red}\bullet$	$\color{red}\bullet$	$\bullet$	$\bullet$	$\color{red}\bullet$	$\color{red}\bullet$	$\bullet$	$\bullet$
Act	$\color{red}\bullet$	$\bullet$	$\color{red}\bullet$	$\bullet$	$\color{red}\bullet$	$\bullet$	$\color{red}\bullet$	$\bullet$	$\color{red}\bullet$	$\bullet$	$\color{red}\bullet$	$\bullet$	$\color{red}\bullet$	$\bullet$	$\color{red}\bullet$	$\bullet$
Red_I	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
Black_I	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
Red_II	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
Black_II	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

Table 3: Decision matrix of the two-colors experiment ("1" stands for \$100).

In the two-colors experiment there are $2^4 = 16$ states, whereas the three-colors experiment contains $3^4 = 81$ states.[8] Each state represents a "strategy" of nature. More precisely, it is a complete list of (actual and hypothetical) "responses" of nature to every act that is available to the decision-maker. In this way, we can assign each act another subjective probability of Red and Black in the two-colors experiment, or Red, Black, and Yellow in the three-colors experiment.

The decision matrix of the two-colors experiment is given by Table 3. State ω_1 means that Red occurs, irrespective of which action the decision-maker performs. State ω_2 describes that the outcome is Black if the decision-maker chooses Black_II, but otherwise he obtains Red, etc. A decision-maker who is ambiguity averse considers $\{\omega_9, \omega_{10}, \omega_{11}, \omega_{12}\}$ more probable than $\{\omega_5, \omega_6, \omega_7, \omega_8\}$. The Ellsberg paradox is simply based on the fact that all elements of Ω except for $\omega_1, \omega_4, \omega_{13}$, and ω_{16} are neglected. Thus, an essential part of Ω is supposed to be null. After reducing Ω to $\{\omega_1, \omega_4, \omega_{13}, \omega_{16}\}$, the decision matrix in Table 3 turns into the decision matrix in Table 1. Of course, if we eliminate some elements of the state space, we cannot properly explain the behavior of a rational decision-maker whose decisions are essentially based on the eliminated states. The same arguments apply to the three-colors experiment.

4. Conclusion

Ellsberg's thought experiments are brilliant. He vividly demonstrates that, in real-life situations, there can be different degrees of uncertainty, and it seems obvious that people prefer less ambiguous alternatives to more ambiguous ones. However, ambiguity aversion can be readily explained by subjectivistic decision theory. The solution to Ellsberg's paradox is simple and underpins the basic idea of subjectivistic decision theory, namely that subjective probabilities reflect the individual preferences of a decision-maker. Those preferences may well be affected by ambiguity aversion. Hence, a nice by-product is that subjective probabilities could even be used in order to quantify the ambiguity aversion of a rational subject.

[8] The exponent, 4, quantifies the number of acts. The base, i.e., 2 or 3, is the number of colors.

References

Ellsberg, D. (1961): "Risk, ambiguity, and the Savage axioms," *Quarterly Journal of Economics* **75**, pp. 643–669.

Savage, L. (1972): *The Foundations of Statistics*, Dover, 2nd edition.