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HELMUT SCHMIDT UNIVERSITÄT
Universität der Bundeswehr Hamburg
Faculty of Economics and Social Sciences
Department of Mathematics/Statistics

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Gabriel Frahm

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Gabriel Frahm

Helmut Schmidt University
Faculty of Economics and Social Sciences
Department of Mathematics/Statistics
Chair for Applied Stochastics and Risk Management
Holstenhofweg 85, D-22043 Hamburg, Germany

URL: www.hsu-hh.de/stochastik
Phone: +49 (0)40 6541-2791
E-mail: frahm@hsu-hh.de

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Supervised by: Prof. Dr. Gabriel Frahm
Chair for Applied Stochastics and Risk Management

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Gabriel Frahm†
Helmut Schmidt University
Department of Mathematics/Statistics
Chair of Applied Stochastics and
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Abstract

Aumann's model of strategic conflict is based on the assumption that strategic uncertainty can be translated into imperfect information. His formal approach suggests that the players are cognizant of each other's strategy and their strategy choices are mutually independent. Aumann's main conclusion is that correlated equilibrium is an unavoidable consequence of Bayesian rationality. I show that cognizance and strategic independence contradict each other. This means that if the players know each other's strategy, they choose their strategies in a coherent way. By contrast, strategic independence can readily be justified if the players completely ignore each other's strategy, i.e., if their decisions are based on belief rather than knowledge. Aumann's model is not able to distinguish between cognizance and ignorance. This problem is resolved by introducing conjecture functions. It turns out that correlated equilibrium is neither necessary nor sufficient for Bayesian rationality. This is demonstrated by means of the prisoners' dilemma, where we can explain both the existence of the cooperative and the noncooperative solution. Hence, correlated equilibrium represents an artifact rather than a natural consequence of Bayesian rationality in strategic games.

Keywords: Belief, conjecture, correlated equilibrium, counterfactual, imperfect information, knowledge, prisoners' dilemma, strategic independence, strategic uncertainty.

JEL Subject Classification: C70, D81.

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† Phone: +49 40 6541-2791, e-mail: frahm@hsu-hh.de.

> "*The logical roots of game theory are in Bayesian decision theory. Indeed, game theory can be viewed as an extension of decision theory (to the case of two or more decision-makers), or as its essential logical fulfillment. Thus, to understand the fundamental ideas of game theory, one should begin by studying decision theory.*"
>
> Myerson (1991, p. 5)

1. Motivation

This work is based on Aumann's model of strategic conflict (Aumann, 1987). In contrast to Aumann, I argue that correlated equilibrium is not an unavoidable consequence of Bayesian rationality in strategic games. I show that Aumann's model leads to rational solutions in which the players choose their strategies in a coherent way. This means that the assumption of strategic independence, which plays a major role for correlated equilibrium, is violated. The problem disappears after a slight extension of the model, which is compatible with Aumann's subjectivistic approach to rational choice under uncertainty. It enables us to explain cooperative and noncooperative behavior of rational players. Further, we can explain the epistemic conditions under which the rational solution of a strategic conflict is a correlated equilibrium. Nonetheless, in the given framework, a rational solution need not be a correlated equilibrium and thus not a Nash equilibrium.[1]

Typical arguments supporting Nash equilibrium are vividly discussed by Risse (2000). He comes to the conclusion that "All of these arguments either fail entirely or have a very limited scope." For example, he argues that

> "*One agent judges deviating reasonable if the other one deviates as well. The other one figures this out. Suppose mutual deviation is profitable for him as well. Since he knows that the first agent would deviate if he did, he deviates. In this way, two agents transparent to each other could abandon an equilibrium. The point is that a Nash Equilibrium, by definition, only discourages uni-lateral deviation.*"

Similarly, Brams (1994, p. 4) mentions that players should "think ahead in deciding whether or not to move." This means that they need to consider "the consequences of a series of moves and countermoves from the resulting outcome." Moreover, Brams (1994, pp. 22–23) elaborates:

> "*The standard theory … does not raise questions about the rationality of moving or departing from outcomes — at least beyond an immediate departure, à la Nash. … The question then becomes whether a player, by departing from an outcome, can do better not just in an immediate or myopic sense but, instead, in an extended or nonmyopic sense.*"

Risse's argument is myopic but allows for multi-lateral deviations, whereas Brams considers alternating moves and countermoves around the payoff matrix of a normal-form game. His

[1] A Nash equilibrium is a correlated equilibrium in which the strategies are stochastically independent.

Theory of Moves (TOM) is based on an extensive-form analysis with backward induction and its goal is to determine where play will end up after starting from any initial state.[2] Thus, even uni-lateral deviations can lead, in the long run, to nonmyopic equilibria that are not predicted by standard game theory. Both authors note that the strategy choices of rational players can depend on each other, i.e., the solution of a game may not be a Nash equilibrium. This is also the key finding of this work. It will be addressed in more detail in the subsequent analysis.

Nash equilibrium has often been considered insufficient or implausible. For example, Colman (2004) points out the possibility of multiple and Pareto-inefficient Nash equilibria (Harsanyi and Selten, 1988). Moreover, Nash equilibria can be subgame imperfect (Selten, 1975) and the idea that the players use a random generator when applying a "mixed strategy" leads to an obscure and nonsensical description of real-life situations (Rubinstein, 1991). Hence, the message that a Nash equilibrium need not be a rational solution of a strategic conflict is not new. Put another way, Nash equilibrium is not *sufficient* for the rationality of all players.

One can find a large number of procedures that aim at a refinement of Nash equilibrium. That is, additional criteria such as Pareto efficiency, subgame perfectness, stability, etc., have been developed to eliminate all Nash equilibria that are considered implausible (see, e.g., Govindan and Wilson, 2008, Harsanyi and Selten, 1988, Kreps and Wilson, 1982, Myerson, 2001, Selten, 1965, 1975). Hence, Nash equilibrium is generally assumed to be a valid but insufficient solution concept. I think that this approach does not address the root of the problem: A rational solution need not be a Nash equilibrium. This means that Nash equilibrium is not even *necessary* for the rationality of all players—and the same conclusion can be drawn for correlated equilibrium.

Bernheim (1984) and Pearce (1984) introduce the notion of rationalizable strategies. They accept solutions that go beyond Nash equilibrium and thus, in this respect, rationalizability is contrary to refinement. However, Bernheim (1984) claims that

> "*In a purely non-cooperative framework, ... the choices of any two agents are by definition independent events; they cannot affect each other.*"

In contrast to this opinion, Aumann (1974, 1987) provides a number of examples in which the action of a player depends on the action of another. Consequently, Brandenburger and Dekel (2014) drop the stochastic-independence assumption of rationalizable strategies. Further, Brandenburger and Friedenberg (2014) distinguish between "intrinsic" correlation, where the belief hierarchies of the players are assumed to depend on each other,[3] and Aumann's "extrinsic" correlation, typically attributed to a physical source of information or signal. In fact, Aumann's model of strategic conflict does not allow only for *stochastic* dependence; this formal approach leads to *strategic* dependence, which will be detailed below.

Classical arguments of noncooperative game theory are based on the assumption of common knowledge, regarding the structure of the game, i.e., the private information partitions of the players, their action sets, subjective probability measures, utility functions, etc., and the players'

[2] I thank Steve Brams very much for clarifying this point in a personal communication.

[3] A belief hierarchy specifies the players' conjectures about each other's strategy, their conjectures about each other's conjecture about each other's strategy, etc. (Perea, 2012, p. 70).

rationality. The notion of common knowledge goes back to Lewis (1969) and has been formalized by Aumann (1976).[4] The key idea is that the players can step into each other's shoes and deduce his optimal strategy—given that each other steps into each other's shoes, etc. This is known as the transparency of reason (Bacharach, 1987). I fear that the common-knowledge assumption is violated in most real-life situations of strategic conflict.

Brandenburger (2014), de Bruin (2010), and Perea (2012) represent a relatively new branch referred to as epistemic game theory. Perea (2012, p. 66) points out that

> "*... we think that the term 'knowledge' is too strong to describe a player's state of mind in a game. ... Player i can at best have a* belief *about the rationality of player j, and this belief may very well be wrong!*"

Hence, the so-called "epistemic program" deals with common *belief* rather than knowledge. Common belief typically refers to the strategies of the players and their rationality. This work goes very much in the same direction. That is, I assume that the players have conjectures about each other and that they are rational. However, the subjectivistic approach adopted here does not require common belief regarding the structure of the game, the strategies of the players, their rationality, or any other personal characteristics.

Every Nash equilibrium is a correlated equilibrium and due to its fundamental importance in economics and many other disciplines, such as biology, psychology, and politics, the results presented in this work might be interesting to a broad audience. Aumann (1987) overcomes several conceptual shortcomings of Nash equilibrium, but there still are some open and highly important questions that will be addressed below. The main contribution of this work is to demonstrate that it is impossible to distinguish between knowledge and belief in Aumann's model, which is crucial if we want to understand the solution of a strategic conflict. The literature on game theory lacks any rigorous, i.e., mathematical, definition of knowledge and belief. This problem is resolved by introducing conjecture functions, which is a convenient way of describing whether somebody is cognizant or ignorant of each other's strategy. The presented approach is astonishingly simple, compared to the "epistemic program," where the assumptions about how players think about each other, how they think that the others think about each other, etc., seem to be quite strong and demanding from a conceptual point of view. We are able to explain solutions that frequently occur in real-life situations in a very simple and intuitive way.

2. Preliminary Remarks

Aumann is a well-known advocate of Savage's (1954) subjectivistic approach to rational choice. I recapitulate the theoretical foundation in the appendix. The reader should be familiar with the given symbols and terminology before moving ahead. However, before I go into the details, I

[4] Let f be any fact and $\kappa(f)$ the fact that f is known to all people. The latter fact is referred to as mutual knowledge. Common knowledge of f means that $\kappa^i(f)$ for all $i \in \mathbb{N}$, i.e., everybody knows f and everybody knows that everybody knows f, etc. For example, if two persons have witnessed a murder and recognized the murderer, he is mutually known. In addition, if the witnesses have seen each other while recognizing the murderer, he is commonly known.

would like to discuss some principal questions. These questions may appear trivial to the reader, but after many discussions about the subject matter, I have come to the conclusion that the following points should be clarified for a better understanding.

Aumann and Dreze (2009) distinguish between games against nature and strategic games. This suggests that the principles of rational choice hold irrespective of whether we suppose that there is only a single decision-maker faced with nature or a number of players competing with each other in a situation of conflict. Hence, each decision-maker may be considered a player and vice versa. Moreover, I make no distinction between "cooperative" and "noncooperative" strategic games. According to Selten (2001) every "noncooperative" game can lead to cooperation and thus may turn into a "cooperative" game—if this is in the interests of each player and the rules of the game allow for cooperation. Hence, contrary to the classical paradigm of game theory, cooperation is viewed as a possible *result* but not as a prerequisite of a strategic game. This means whether players cooperate or not should follow from the formal description of the game and not determined from the outset. This forces us to develop a methodological framework that is general enough to allow for both cooperative and noncooperative behavior. I think that this goal has been accomplished in this work. This is demonstrated by means of the prisoners' dilemma. I do *not* assume that cooperative players are altruistic. Whenever cooperation occurs, it is a consequence of the rational behavior of egoistic players.

Aumann's model of strategic conflict consists of the following primitives (Aumann, 1987, p. 3):

- The number of players, i.e., $n \in \mathbb{N} \setminus \{1\}$.
- A measurable space $(\Omega, \mathscr{F})$, where Ω denotes the state space and $\mathscr{F}$ is a σ-algebra.
- The private information partitions $\mathscr{I}_1, \mathscr{I}_2, \ldots, \mathscr{I}_n$ of the players.
- Their action sets $A_1, A_2, \ldots, A_n \subseteq \mathscr{A}$.
- Their subjective probability measures $p_1, p_2, \ldots, p_n$.
- Their utility functions $u_1, u_2, \ldots, u_n$.
- Their possible strategies $s_1, s_2, \ldots, s_n$, where Strategy s_i is a function from Ω to A_i that is constant on each element of $\mathscr{I}_i$ $(i = 1, 2, \ldots, n)$.[5]

Aumann's model describes strategic conflicts in their normal form. The distinction between normal-form and extensive-form games is not understood in the generic sense. This means that I do not put any strategic game into the category "normal-form game" or "extensive-form game." Each game can be represented both in normal form and in extensive form. Since it is always possible to represent a repeated game in its normal form, we do not have to restrict ourselves to "one-shot games." For example, the game of chess is usually represented in extensive form. Nonetheless, in the general framework established above, it could also be represented in normal

[5] In the following, the enumeration "$i = 1, 2, \ldots, n$" will be omitted whenever it seems clear from the context that the corresponding statement applies to each player.

form, where the action of a chess player is an exhaustive plan of moves and countermoves. In decision theory, such a plan is often called a "strategy." Unfortunately, an abuse of terminology cannot be completely avoided when switching between game and decision theory.

Throughout this work, I refer only to the normal form of strategic games. This has some important consequences regarding the basic interpretation of the model. In the normal form of a game there is no time dimension. Thus, at least in a formal sense, actions and consequences cannot be associated with any point in time. Moreover, due to the lack of the time dimension, we cannot say that the players choose their strategies simultaneously. The widespread idea that players in a normal-form game choose their strategies at the same time shall only express a typical assumption in noncooperative game theory, namely that the strategy choice of a player does not depend on the strategy choice of any other player. This means that the aforementioned assumption of strategic independence is satisfied. It plays a major role in this work and will be discussed in more detail below.

We may also conclude that the sentence "Player i applies the strategy s_i and Player j responds with the strategy s_j" lacks any chronological meaning. This means that Player j cannot respond to Player i *after* the latter has made his decision. Otherwise we would have had to formulate the game in extensive form before translating it into its normal form. In the normal form, no player can respond to another in the *chronological* sense. Nonetheless, the players might be able to respond to each other in a *logical* sense. For example, Player j could choose the strategy s_j *because* Player i chooses the strategy s_i. If we take this possibility into account, we must drop the strategic-independence assumption. I will come back to this crucial point later.

The strategy of a player is nothing other than a random variable. Probability theory deals with random variables that are considered *given*. Hence, the probabilistic viewpoint reflects the situation after the players have made up their minds and so it is *ex post*. By contrast, here we want to analyze the procedure of rational choice, where each player evaluates his possible alternatives before he comes to a final conclusion, i.e., before the random variable exists. This means that we must consider the situation before the player has made his decision and so the decision-theoretic viewpoint is *ex ante*. By distinguishing between the different points of view we can avoid serious misunderstandings that can arise when discussing situations of strategic conflict.[6] Aumann (1987, p. 7) calls a player (Bayes) rational "if his expected payoff given his information is at least as great as the amount that it would have been had he chosen an action other than the action that he did in fact choose." In principle, I have no objections to that. However, Aumann's formal treatment suggests that the choice of the player has no influence on the choice of any other player. This may create the impression that we adopt the probabilistic, i.e., ex-post, point of view, but this is not true. Aumann (1995) points out that

> "*Making a decision means choosing among alternatives. Thus one* must *consider hypothetical situations—what would happen if one did something different from what one actually does.*"

Similarly, Stalnaker (1996) writes that

[6]The different viewpoints have nothing to do with the notion of "prior" and "posterior" in Bayesian decision theory.

> "*Deliberation about what to do in any context requires reasoning about what will or would happen in various alternative situations, including situations that the agent knows will never in fact be realized.*"

Thus, since we want to analyze the strategic behavior of the players, we must adopt the decision-theoretic, i.e., ex-ante, point of view. If we take the perspective of Player i, in fact we may consider the strategies of the others fixed *after* they have made their decisions, but then Player i has already made up his mind, too. This means that the whole procedure of rational choice is already completed. This would not serve our purpose. By contrast, if we adopt the ex-ante point of view for Player i, we cannot at the same time adopt the ex-post point of view for any other player. Why, then, should it be possible to consider the strategies of the others fixed, while Player i still may choose between his possible alternatives? This essential question will be elaborated in the subsequent analysis.

Throughout this work, I adopt the ex-ante point of view whenever I compare the available strategies $s_1, s_2, \ldots, s_n$ of the players with each other. This point of view is purely *hypothetical*, i.e., it occurs only in the mind of a player or game theorist. For example, the statement "If Player i chooses the strategy s_i, Player j responds with the strategy s_j" or, in short, "$s_i \Rightarrow s_j$" represents a substantive but not a material conditional (Aumann, 1995). This means that I ignore the actual choice of Player i and simply *imagine* that Player i chooses the strategy s_i, which leads to the response s_j of Player j—provided the substantive conditional $s_i \Rightarrow s_j$ is true. If, in fact, Player i chooses another strategy, the substantive conditional is said to be a counterfactual (Lewis, 1973). For a material conditional "$A \Rightarrow B$" to be true it is sufficient to show that its antecedent, A, is false. By contrast, for a counterfactual "$A \Rightarrow B$" to be true we must guarantee that its consequent, B, *would* have been true if A *were* true.

By contrast, whenever I consider the solution of a game, i.e., the strategies that have *actually* been chosen by the players, I adopt the ex-post point of view. This point of view is *factual* rather than hypothetical and the corresponding strategies are denoted by $s_1^*, s_2^*, \ldots, s_n^*$. In this situation, nobody can substitute his strategy with another one, since the strategic conflict is already solved. Put another way, the game is over. In general, I accept the solution of a game as it is and do not require that it is in any sense rational. This means that the term "solution" should always be understood in a descriptive rather than prescriptive sense.

To understand the following arguments, it is crucial to distinguish between states and actions. A state, ω, is an (abstract) element of the state space Ω without any substantive meaning. Each state of the world, $\omega \in \Omega$, leads to a certain action $a_i \in A_i$ of Player i, which depends on his chosen strategy s_i.[7] More precisely, if the state $\omega \in \Omega$ obtains, Player i performs the action $a_i = s_i(\omega)$. Aumann (1987, p. 6) mentions that "each ω includes a specification of which action is chosen by each player of G at that state ω." However, the action of Player i at the state ω is determined by s_i. Thus, even if somebody knows ω, he cannot be aware of the action of Player i, at the state ω, if s_i is unknown to him. Indeed, it is true that ω implies the action of Player i, but only if we specify his strategy s_i, which assigns a certain action to each state of the world.

[7] Hence, the object of choice is s_i but not ω. I will come back to this essential point later.

Table 1: Prisoners' Dilemma

Two gangsters have committed a crime and have been arrested. Apart from illegal possession of arms, there is no evidence against them. Now, each prisoner can either deny or confess. If both prisoners deny, they serve only one year in prison. If one player denies and the other confesses, the latter is set free as a principal witness, whereas the former is sentenced to five years. Moreover, if both players confess, each of them serves four years in prison.

Now, assume that Player i and Player $j \neq i$ share the same private information partition, i.e., $\mathscr{I}_i = \mathscr{I}_j$. Does Player i know the action of Player j at each state of the world $\omega \in \Omega$? Well, this crucially depends on whether Player i knows the strategy of Player j. Indeed, Player i is informed about $I_i \in \mathscr{I}_i$ that is such that $\omega \in I_i$. Moreover, since the players share the same private information partition, the strategy of Player j is constant on I_i. Hence, Player i knows the action of Player j, for all $\omega \in \Omega$, if and only if he knows Player j's strategy s_j. The problem is that the private information partition of Player i, $\mathscr{I}_i$, does not say anything about the question of whether he knows the strategy of Player j or not. Thus, it can happen that the players do *not* know each other's action although they have the same private information partitions. This is a key observation of this work.

For example, consider the prisoners' dilemma (Table 1). The number of years that each prisoner must spend in jail is given by the following payoff matrix:

	Joe	
Mary	Deny	Confess
Deny	$(1,1)$	$(5,0)$
Confess	$(0,5)$	$(4,4)$

The labels "deny" and "confess" do not represent states of the world. Rather, the rows and columns of the given payoff matrix contain the possible actions of the players.[8] We assign each possible action a real number: 0 stands for "deny" and 1 for "confess." For the sake of simplicity, we assume that Mary (Player 1) and Joe (Player 2) can choose only between "pure strategies." This assumption refers to the private information partitions of the players. We could also have chosen a model that allows for "mixed strategies," but this would not have any substantial impact on our conclusions. If Mary wants to deny, we obtain $s_1(\omega) = 0$ for all $\omega \in \Omega$. By contrast, if she plans to confess, we have that $s_1(\omega) = 1$ for all $\omega \in \Omega$. The same principle applies to Joe. Thus, Mary and Joe share the same private information partition $\mathscr{I}_1 = \mathscr{I}_2 = \{\Omega\}$. Nonetheless, Mary is *not* aware of the action of Joe—unless she knows his strategy, and the same holds true, mutatis mutandis, for Joe.

There exists another source of misunderstanding. I think that this is the most controversial issue in Aumann's model of strategic conflict: It is often suggested that $\omega \in \Omega$ is a control variable. In fact, Aumann (1987, pp. 8–9) mentions that "the decision taken by each decision maker is part of the description of the state of the world" and "That we include his action as part of the

[8] Hence, the given payoff matrix is not a decision matrix, where each column represents a state of nature.

description of the state of the world is only a convenient way of expressing the fact that the other players do not know which action he wishes to choose." Hence, the argument might go like this: "Player i suffers from imperfect information. This means that Player j chooses some state $\omega \in \Omega$, which leads to his action $s_j(\omega) \in A_j$, but Player i does not know which ω obtains and so he is not aware of the action of Player j." In my opinion, the entire argument fails: The state of the world cannot be a control variable, i.e., Player j cannot *choose* $\omega \in \Omega$. Indeed, we assume that players have a free will, but if everyone could simply decide which state of the world obtains, Player i might choose some state ω_i, whereas Player $j \neq i$ could choose another state $\omega_j \neq \omega_i$. Here, ω_i and ω_j belong to the *same* state space Ω and thus we could obtain n different outcomes of the game, i.e., $\omega_1, \omega_2, \ldots, \omega_n \in \Omega$. This would reduce Aumann's model of strategic conflict to absurdity. According to my basic understanding of subjectivistic decision theory, no player can arbitrarily choose $\omega \in \Omega$. Indeed, the fact that players are *not* able to control $\omega \in \Omega$ is precisely the reason that they assign each event $F \in \mathscr{F}$ a subjective probability (Savage, 1954).

In this context, Levi (1997a, p. 81) coins the phrase "deliberation crowds out prediction." The same argument is raised, among others, by Spohn (1977).[9] Hence, the state of the world, ω, cannot be an object of choice—irrespective of whether we argue from the perspective of a single decision-maker in a game against nature or a player in a strategic game. Aumann (1987, pp. 2–4,7) clarifies that the objects of choice are the strategies, $s_1, s_2, \ldots, s_n$, rather than the states of the world, i.e., the elements of Ω.[10] More precisely, according to Aumann (1974), his model of strategic conflict involves the following steps:

(i) Nature chooses the state of the world, i.e., $\omega \in \Omega$.

(ii) Player i is informed about $I_i \in \mathscr{I}_i$ that is such that $\omega \in I_i$.

(iii) Now, based on his private information, Player i performs some action $a_i \in A_i$.[11]

It is assumed that Player i assigns each element of his private information partition an action *before* he is informed about $I_i \ni \omega$ (Aumann, 1974, p. 74) or even before nature chooses $\omega \in \Omega$. In this way, Player i implicitly creates or, say, "chooses" a function s_i from Ω to A_i that is constant on each element of $\mathscr{I}_i$—which is referred to as his strategy. Thus, in Aumann's model of strategic conflict, deliberation and prediction refer to different objects and so they do *not* contradict each other. It seems to me that the longstanding controversy about the possible interpretations of Aumann's model can be attributed, in large part, to the following causes: (i) "State" is often used, incorrectly, as a synonym for "action" or "strategy," and (ii) it is often assumed, either implicitly or explicitly, that $\omega \in \Omega$ is an object of choice.[12] These sources of confusion go hand in hand.

Further, Aumann (1987, p. 2) states that "it is common knowledge that all the players are Bayesian utility maximizers, that they are rational in the sense that each one conforms to the

[9] See the references in Bonanno (2015, fn. 25) for more details on that topic.

[10] Even though we do not agree on this point, I thank very much Giacomo Bonanno for discussing the question of whether or not players can choose their strategies. In my opinion, the notion of rational choice makes sense only if each player is able to choose his own strategy.

[11] Aumann (1974, 1987) calls a_i a "pure strategy."

[12] For example, saying that ω contains the state of mind of a player suggests that he is able to choose deliberately between the elements of Ω, i.e., that the state of the world is an object of choice.

Savage theory." This statement is somewhat misleading. In fact, Aumann (1987, p. 10) points out that common-knowledge assumptions only "aid us in understanding the model, they do not affect the conclusions." Actually, we do not scrutinize the strategic reasoning of the players, i.e., we do not ask whether a rational choice is in any sense reasonable (Perea, 2012, p. 65). Our subjectivistic approach is descriptive rather than prescriptive. For this reason, it is not necessary to assume that players have a belief hierarchy and common belief in rationality. Thus, I refrain from common belief, which is a typical assumption in epistemic game theory.

3. Cognizance vs. Ignorance

3.1. Cognizance

Aumann (1987) transfers Savage's basic principles of rational choice to strategic conflicts. He mentions that "The chief innovation in our model is that it does away with the dichotomy usually perceived between uncertainty about acts of nature and of personal players." This means that each player may think of the conflict as a game against nature where "nature" is represented by the other players. Aumann overcomes several conceptual shortcomings of Nash equilibrium (Nash, 1951). For example, he resolves the randomization paradox (Rubinstein, 1991) and is able to describe a typical situation of strategic conflict—namely that players have differential information. In Aumann's model of strategic conflict, a rational player considers the strategy of each other player a state variable, but Aumann omits making a trivial but important observation: A rational player must not ignore the fact that each other player considers his own strategy a control variable. This can lead to quite complicated strategic interactions that typically do not appear in nature. In my opinion, this distinguishes games against nature from strategic conflicts.

This is also the core idea of TOM (Brams, 1994), which thrives on the fact that rational players should think ahead when choosing a strategy (Brams, 1994, fn. 3): "If players do begin by choosing strategies, I assume that they can anticipate subsequent moves in the matrix ..." Hence, Brams (1994, fn. 11) allows the moves and countermoves to occur in the minds of the players, but he actually uses the normal form just to explain *dynamic* behavior:

> "*... players may be thought of as choosing strategies initially, after which they perform a thought experiment of where moves will carry them once a state is selected. ... Generally, however, I assume that 'moves' describe actions, not just thoughts, though I readily admit the possibility of the thought interpretation.*"

In this work, the strategic interactions that occur in the normal form of a game are only *ex ante*, i.e., they are part of the strategic reasoning of the players but do not reflect their actual choices.

In his seminal work, Savage (1954, p. 15) treats an important aspect of subjectivistic decision theory:

> "*The argument might be raised that the formal description of decision ... seems inadequate because a person may not know the consequences of the acts open to him in each state of the world.*"

According to Savage it should always be possible to cut each element of Ω into pieces, i.e., to dissect every potential source of ambiguity, until every state of nature, ω, leads to one and only one consequence $s(\omega)$.[13] The Savage act, s, may depend on the action of the decision-maker, but once his action and the state of nature, ω, are fixed, he cannot be uncertain about $s(\omega)$. Put another way, it is assumed that the state space, Ω, is properly specified.[14] According to Savage, the decision-maker might not know ω, but if somebody tells him ω, he certainly *knows* $s(\omega)$, i.e., the consequence of his decision that occurs if $\omega \in \Omega$ obtains.

Bayesian rationality is typically considered part of the subjectivistic approach to rational choice. Aumann (1987, p. 2) concludes that

> "*According to the Bayesian view, subjective probabilities should be assignable to every prospect, including that of players choosing certain strategies in certain games.*"

Thus, it should always be possible to translate strategic uncertainty into imperfect information. This view is also held by Aumann and Dreze (2009) and the same principle can be found in Harsanyi's (1967–1968) pioneering work on games with incomplete information. That is, in Aumann's model of strategic conflict, the players do *not* suffer from strategic uncertainty, but they may have imperfect information—and the private information partition $\mathscr{I}_i$ precisely reflects the amount of information of Player i about the state of the world $\omega \in \Omega$.

According to Aumann (1987, p. 6),

> "*The term 'state of the world' implies a definite specification of all parameters that may be the object of uncertainty on the part of any player of G. In particular, each ω includes a specification of which action is chosen by each player of G at that state ω. Conditional on a given ω, everybody knows everything; but in general, nobody knows which is really the true ω.*"

This means that the players might not know the state of the world, ω, but if ω were revealed to them, they would certainly *know* each other's action. This fits perfectly with Savage's view of games against nature. However, against this backdrop, some statements in Aumann (1987) could lead to misunderstandings. He points out that "Nash equilibrium does make sense if one starts by assuming that, for some specified reason, each player knows which strategies the other players are using. But this assumption appears rather restrictive." Further, he mentions that "in our treatment, the players do *not* in general know how others are playing."

Despite the last statement, the aforementioned arguments suggest that the players *know* each other's strategy. Indeed, after strategic uncertainty has been translated into imperfect information, the players may be uncertain about each other's action, but they must be certain about the others' strategies. Otherwise, some player would suffer from strategic uncertainty—which can be translated into imperfect information. Aumann's approach to strategic conflict not only implies that the players are certain about each other's strategy but, according to his

[13] See the omelet example in Savage (1954, pp. 13–15).
[14] Cutting the state space into pieces until it is properly specified could be referred to as a refinement of Ω.

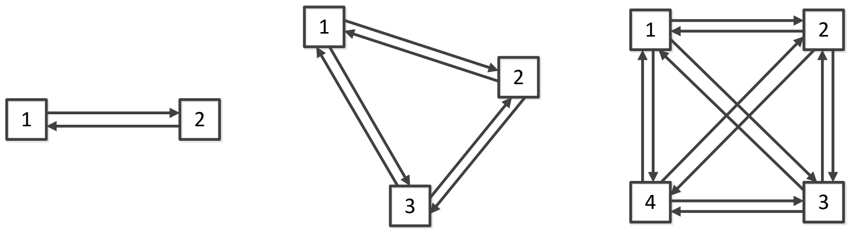

Figure 1: Influence diagrams for a 2- (left), 3- (middle), and 4-person game (right).

own arguments, that they even *know* which actions take place if the state of the world, ω, were revealed to them, and this holds for all $\omega \in \Omega$. That is, the players know each other's strategy. In fact, the overall distinction between perfect and imperfect information requires the players to be cognizant of each other's strategy. Under these circumstances, a player is said to have perfect information if and only if he knows each other's action.

For example, consider a 2-person game. Suppose that Player 1 applies the strategy s_1 and let s_2 be the response of Player 2. What happens if Player 1 applies another strategy $s_1' \neq s_1$? Well, in this case, Player 2 *knows* that Player 1 applies s_1' instead of s_1. Why should Player 2 still apply s_2 if Player 1 applies another strategy? Put another way, why should the choice of Player 2 be independent of the choice of Player 1? Since Player 2 knows what Player 1 is going to do, he could adapt his own strategy to the strategy of Player 1—and the same holds true for Player 1.

Hence, if the players know each other's strategy, their strategy *choices* typically depend on each other. Here, I am not speaking about stochastic dependence, where the strategies are already given, but the players' actions, i.e., the realizations of $s_1, s_2, \ldots, s_n$, are connected with each other in the usual sense of probability theory. Strategic dependence means that the players choose their strategies in a coherent way. More precisely, each strategy is an implicit function of the other strategies, which is illustrated by the influence diagrams in Figure 1. This notion of "dependence" goes beyond probability theory.

From the perspective of Player i, his own strategy is a control variable, whereas the strategy of each other player is a state variable. According to Aumann's model, the players are cognizant of each other's strategy. Hence, the expected utility of Player i reads

$$\mathrm{E}_i\Big(u_i\big(s_1(s^1), s_2(s^2), \ldots, s_i, \ldots, s_n(s^n)\big)\Big), \qquad \forall\, s_i \in \mathscr{S}_i,\ i = 1, 2, \ldots, n.$$

Obviously, $s_j(s^j)$ may depend on s_i and thus, in general, Player i will *not* consider the other strategies fixed when maximizing his expected utility. This is a blind spot in Aumann's model. The problem is that cognizance and strategic independence contradict each other. If everyone knows each other's strategy, the assumption of strategic independence is typically violated and, under these circumstances, we will usually not obtain a correlated equilibrium.

This can be clarified by the prisoners' dilemma: We may assume that the prisoners are sitting together with their lawyer, who is instructed to announce their testimonies in court. He asks the prisoners to make their choices. In this situation, each player is able to verify the action of the other. Suppose that Player 1 proposes Action 0 ("deny") but Player 2 replies with Action 1 ("confess"). Why should Player 1 accept the reply of Player 2? It is clearly better for him to change his mind and propose Action 1 instead.[15] Now, Player 2 knows that Player 1 chooses Action 1 if he chooses Action 1. What happens if Player 2 proposes Action 0 instead? In the same way, we could argue that the players end up with Action 1, i.e., with the noncooperative solution of the prisoners' dilemma. Is this solution plausible—given that the players *know* the other's strategy? I think that the answer is "No!" The players could simply do better by playing "tit for tat" (Axelrod and Hamilton, 1981): One player defects if and only if the other defects. This means that the strategy choices depend on each other and in this case "confess" does not dominate "deny." The keynote is that the players are cognizant of the other's response and so they cannot betray each other. Thus, we obtain the cooperative solution of the prisoners' dilemma. Here, cooperation is not a prerequisite of the game. It rather occurs as a result of our principal assumption that the players know each other's strategy, i.e., that they are cognizant. Whether there exists a binding agreement or not is only of secondary importance. It is relatively easy to find examples in which the players are cognizant without a binding agreement. Moreover, our conclusions do not change if we drop the assumption of perfect information but still assume that the players are cognizant. In this case, a player does not know the other's action but, nonetheless, both players know the other's *strategy* and so they are still able to choose their strategies in a coherent way.

How can we distinguish between knowledge and belief without leaving Aumann's model? Players always have conjectures about each other. Our subjectivistic approach to rational choice does not require the conjectures of the players to be correct but, since the players do not suffer from strategic uncertainty, their conjectures at least must be unambiguous. This means that the players must be convinced about their opinion regarding the strategy of each other player. Otherwise, the state space, Ω, would not be properly specified. For this reason, we can assume that each player has a conjecture *function* and not just a correspondence, which would allow the players to have multiple conjectures about each other. Since Player i considers his own strategy a control variable, whereas the strategy of each other player represents a state variable, Player i must have a (unique) conjecture about each other's strategy *for every possible choice of* s_i.

Recall that $\mathscr{S}_i$ is the set of all strategies that are available to Player i and $\mathscr{S}$ denotes the set of all strategy tuples $s = (s_1, s_2, \ldots, s_n)$ that are possible in the game. Further, $\mathscr{C}$ is the set of all functions from Ω to $\mathscr{A}^n$ (see Section A.2). Our first definition is a building block of this work:

Definition 1 (Conjecture function)**.** *The* conjecture function *of Player i is given by*

$$\begin{aligned} \psi_i : \mathscr{S}_i \quad &\longrightarrow \quad \mathscr{C} \\ s_i \quad &\longmapsto \quad \Psi_i\big(s_1, s_2, \ldots, s_n\big), \end{aligned}$$

where $\Psi_i = \big(\Psi_{i1}, \Psi_{i2}, \ldots, \Psi_{in}\big)$ is a function from $\mathscr{S}$ to $\mathscr{C}$ such that $\Psi_{ii}(s) = s_i$ for all $s \in \mathscr{S}$.

$\psi_i(s_i)$ represents the conclusion that Player i has made about (his own and) each other's strategy. More precisely, $\psi_{ij}(s_i)$ is the conjecture of Player i about the strategy of Player j—given

[15] Note that the given situation is still *ex ante*.

that Player i decides to choose the strategy s_i. Thus, $s_i \Rightarrow \psi_{ij}(s_i)$ represents a substantive conditional that is not necessarily true but, however, Player i at least *believes* that it is true.

The overall approach can be decomposed into an objective and a subjective part: The *true* substantive conditional $s_i \Rightarrow (s_1, s_2, \ldots, s_n)$ transforms the strategy of Player i into the strategy tuple $(s_1, s_2, \ldots, s_n)$, whereas the map Ψ_{ij} transforms the strategy tuple into the conjecture of Player i about the strategy of Player j. So far, I do not presume that the strategy of any player is a "best response" or, in any other sense, optimal. Moreover, I do not suppose that Player i is aware of the true relationship between the strategies $s_1, s_2, \ldots, s_n$. The crucial point is that, in some specific situations of strategic conflict, the conjectures of a player might depend on the strategies of the other players. This special feature of the conjecture function is used in the subsequent analysis to distinguish between knowledge and belief.

A similar approach can be found in Board (2004), Bonanno (2015), Gibbard and Harper (1978), Shin (1992), Stalnaker (1996), and Zambrano (2004). However, to the best of my knowledge, the aforementioned authors do not allow the conjectures of a player to depend on each other's strategy. For example, Stalnaker (1996) writes that

> "… *the assumption is that the strategies are chosen independently, which means that the choices made by one player cannot influence the beliefs or the actions of the other players.*"

Moreover, epistemic game theory usually presumes that the conjectures of a player about the others, i.e., his first-order beliefs (Perea, 2012, p. 80), not even depend on his *own* strategy. Hence, the approach chosen here differs essentially from the typical approach of epistemic game theory: I allow the conjectures of a player to depend on his own strategy and on the strategies of all other players. Further, I do not assume that the players have complete information, i.e., that Player i knows the private information partition, action set, etc., of Player j for all $j \neq i$, and I do not require common belief in rationality.

Now, it is time to give a formal definition of the philosophical term "cognizance."

Definition 2 (Cognizance)**.** *Player i is said to be* cognizant of the strategy of Player $j \neq i$ *if and only if*

$$\Psi_{ij}(s) = s_j, \qquad \forall\, s \in \mathscr{S}.$$

In particular, Player i is said to be cognizant *if and only if he is cognizant of each other's strategy.*

If Player i is cognizant, Ψ_i is an identity, i.e., we have that $\Psi_i(s) = s$ for all $s \in \mathscr{S}$ and thus it holds that $\psi_i(s_i) = s$ for all s_i. Put another way, a cognizant player knows the strategies of the others—*irrespective of their particular choices.* Hence, for each available strategy $s_i \in \mathscr{S}_i$, $s_i \Rightarrow \psi_i(s_i)$ represents a *true* substantive conditional.

Suppose that Player i chooses the strategy s_i and Player $j \neq i$ responds with the strategy s_j. Then the conjecture of Player i about Player j is s_j. By contrast, if Player j gives another response $s_j' \neq s_j$ to s_i, the conjecture of Player i about Player j is s_j'. Thus, for the notion of cognizance, it does not suffice to require the conjectures of a player to be correct only for the *actual* choice of

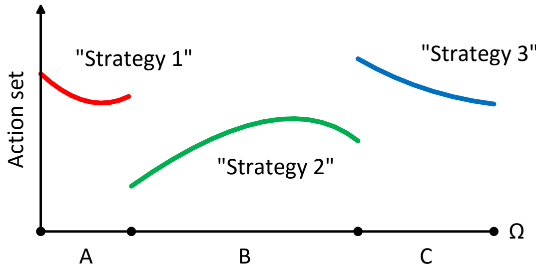

Figure 2: Naive interpretation of Aumann's model.

strategies; they must also be correct for each other *hypothetical* choice of strategies. For example, suppose that the equality $\Psi_i(s^*) = s^*$ is true, where s^* represents the solution of the game. This equality says only that the conjectures of Player i about the other players are correct—given the particular solution of the game. However, it is not meant to say that his conjectures would have been correct if any Player $j \neq i$ *had* chosen another strategy.

One could argue that the aforementioned statements are irrelevant, since the players should be interested only in the actual choices of their opponents. The problem with this argument is that the choice of one player can depend on the choice of another and our whole theory of rational choice is based on the comparison of alternatives or, to say it in Aumann's (1995) own words, of counterfactuals. Hence, for deriving the solution of a strategic conflict, we must properly distinguish between knowledge and belief, which is done in the following section.

3.2. Ignorance

In most real-life situations of strategic conflict the players cannot know each other's strategy. Indeed, their conjectures might be correct for the particular solution of the game, but this fact alone does not guarantee that their conjectures are still correct if anyone departs from the given solution. For example, suppose that there is *no* lawyer in the prisoners' dilemma. In this case, the prisoners cannot know the action of the other. This means that they are not cognizant of the other's response and so their conjectures may be incorrect.[16] Communication would not solve the basic problem, namely that the players are not able to verify the true intentions of the other. Irrespective of whatever one makes the other believe, each player remains *ignorant* of what the other is actually going to do. Hence, everybody can defect without consequences and, under these circumstances, it is always best for the players to confess. This leads to the well-known noncooperative solution of the prisoners' dilemma. Our conclusions do not change if we assume that $\Psi_1(s)$ and $\Psi_2(s)$ depend on $\omega \in \Omega$, i.e., if speak about "strategy" rather than "action." Due to the same arguments, the players cannot be cognizant of the other's *strategy*.

Hence, the typical situation of strategic conflict, where the players are not aware of each other's strategy, cannot be described by Aumann's model. The problem is that we are not able to distinguish between knowledge and belief, i.e., cognizance and ignorance, if we follow Aumann's

[16]We assume only for the sake of simplicity that $\Psi_1(s)$ and $\Psi_2(s)$ are constant on Ω for every $s \in \mathscr{S}$. This means that the players think about the *action* rather than the strategy of the other.

formal approach to strategic conflict. The reader might rightly ask why we cannot simply reduce ignorance to imperfect information. He could raise the following argument: "Player i does not know the choice of Player $j \neq i$. Suppose, for the sake of simplicity, that his private information partition is given by $\mathscr{I}_i = \{\Omega\}$ and consider any partition $\{A, B, C\}$ of the state space Ω. Now, Player j can decide whether to choose Strategy 1, i.e., $\omega \in A$, Strategy 2, i.e., $\omega \in B$, or Strategy 3, i.e., $\omega \in C$. Since Player i has imperfect information, he assigns each event $A, B, C \in \mathscr{F}$, i.e., each possible strategy, a subjective probability." This appealing argument is illustrated in Figure 2. In fact, Aumann (1987, p. 2) concludes that

> "*Rather than playing an equilibrium, the players should simply choose strategies that maximize their utilities given their subjective distributions over the other players' strategy choices.*"

Unfortunately, the aforementioned argument contains two fundamental mistakes:

(a) The state of the world, ω, is not an object of choice. This means that Player j cannot choose Strategy 1, 2, or 3 by choosing an appropriate element from the state space Ω.

(b) There are no different strategies at all. Each graph in Figure 2 is part of one and only one strategy, which covers the entire state space Ω.

Thus, for conceptual reasons, players cannot have any "subjective distributions over the other players' strategy choices." Their subjective probabilities refer to events, i.e., to the elements of the σ-algebra $\mathscr{F}$, but an event cannot be a strategy. Otherwise, each player could simply decide which event is going to happen, which makes no sense in our subjectivistic framework of rational choice under uncertainty. For this reason, we are not able to reduce ignorance, i.e., the fact that a player does not know the others' strategies, to imperfect information.

Does an ignorant player suffer from strategic uncertainty? In Aumann's model of strategic conflict, no player can suffer from strategic uncertainty. According to my own understanding, "strategic certainty" just means that the players *believe* that they know each other's strategy. This is weaker than Aumann's and Savage's notion of certainty, which requires *knowledge* rather than belief. Epistemic game theory deals with conjectures rather than certitudes and so I follow the same paradigm. Here, the notion of "strategic certainty" describes only the fact that a player is *convinced* about his opinion regarding the strategy of each other player; it does not require his conjectures to be correct. Recall that the players are convinced about each other's strategy if and only if the state space, Ω, is properly specified.

Now, we can use the function Ψ_i to give a precise definition of ignorance:

Definition 3 (Ignorance)**.** *Player i is said to be* ignorant of the strategy of Player $j \neq i$ *if and only if*

$$\Psi_{ik}(s) = \Psi_{ik}\big(s_1, s_2, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n\big), \qquad \forall\, s \in \mathscr{S},\ k \neq i\,.$$

In particular, Player i is said to be ignorant *if and only if he is ignorant of each other's strategy.*

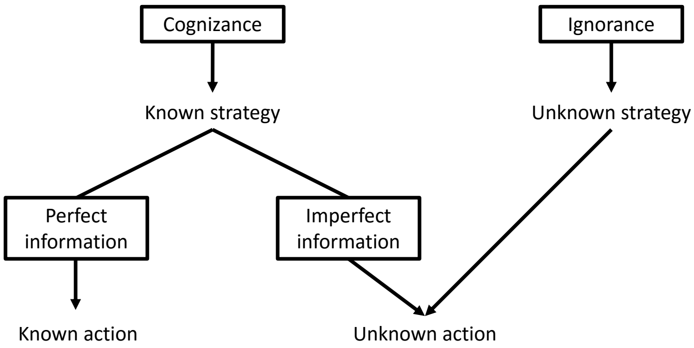

Figure 3: Cognizance and ignorance vs. perfect and imperfect information.

Hence, if Player i is ignorant, we have that $\Psi_i(s_i, s^i) = \Psi_i(s_i)$ for all $s \in \mathscr{S}$. This means that the conjectures of Player i are invariant under a change of s^i, which contains the strategies of the other players (see Section A.2). Indeed, Player i could believe that Player $j \neq i$ is cognizant. In this case, $\Psi_{ij}(s_i, s^i)$ can change with s_i, but his own conjecture about Player j cannot change with s_j—since he is ignorant of Player j's strategy.

The function Ψ_i can be viewed as an eye patch of Player i that is open if he is cognizant and closed if he is ignorant. Put another way, a cognizant player considers the strategies of his opponents *evident*, whereas an ignorant player forms his opinion about each other only by *imagination*. For this reason, he cannot be aware of the others' strategies. A cognizant player can suffer from imperfect information, i.e., he also may not be aware of the action of another player. However, this is only because he does not know which state of the world obtains, not because the other's strategy is unknown to him. That is, the reasons for not knowing the action of another player are entirely different: Players suffer from imperfect information because they have different information about the state of the world. By contrast, if the players are ignorant, they cannot observe any other's strategy, i.e., the actions that their opponents assign to each $\omega \in \Omega$. Hence, ignorance does not come from any lack of private information. In fact, players can be ignorant even if their private information partitions coincide. The connection between cognizance and ignorance as well as perfect and imperfect information is depicted in Figure 3.

The essential difference between cognizance and ignorance is illustrated by the shell game in Figure 4. For the sake of simplicity it is assumed that all strategies and conjectures are constant on Ω. The ball can be only under the left, middle or right shell, and so the player can choose between "left," "middle," and "right." A cognizant player *knows* the right place of the ball. This means that if the thimblerigger chooses "left," the player's conjecture is "left," if he chooses "middle," the player's conjecture is "middle," etc., which can be seen on the top of Figure 4.[17] The point is that the conjecture of a cognizant player is always correct—irrespective of the particular choice of the thimblerigger. By contrast, an ignorant player is only *convinced* about the place of the ball. The crucial point is that the player himself cannot distinguish between "knowledge" and "conviction." Of course, he might be correct for a particular choice of the thimblerigger.

[17] The term "conjecture" could be misleading. It is used only to clarify that the player's opinion is based upon his conjecture function, irrespective of whether he truly knows or just believes that the ball is under a certain shell.

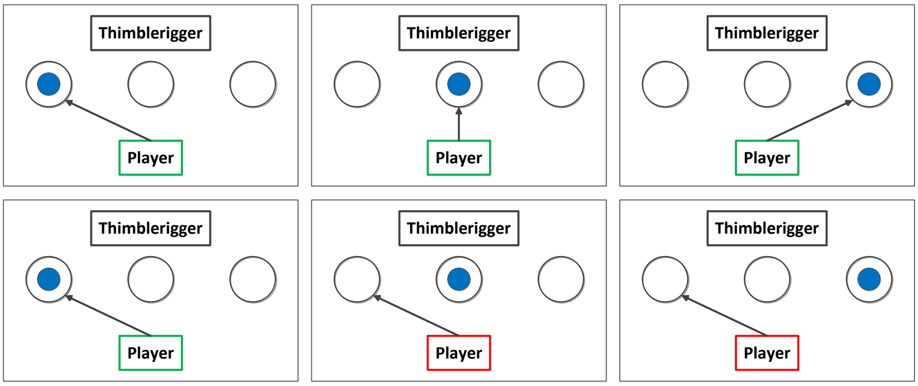

Figure 4: Shell game with a cognizant (top) and an ignorant (bottom) player.

This is illustrated on the lower left of Figure 4. However, his conjecture does not depend on the actual choice of the thimblerigger. If the latter places the ball under another shell—the player's conjecture turns out to be wrong, which is depicted on the lower middle and right of Figure 4.

We can imagine situations in which Player i is ignorant only of some part of the strategy of another player. In such a case we can apply the following definition:

Definition 4 (Partial ignorance)**.** *Player i is said to be* partially ignorant of the strategy of Player *$j \neq i$ if and only if there exists an event $F_j \in \mathscr{F}$ with $F_j \neq \Omega$ such that*

$$\Psi_{ik}(s) = \Psi_{ik}\big(s_1, s_2, \ldots, s_{jF_j}, \ldots, s_n\big), \qquad \forall\, s \in \mathscr{S},\, k \neq i\,.$$

A cognizant player cannot be partially ignorant, but an ignorant player is always partially ignorant, which can be seen by setting $F_j = \emptyset$ in Definition 4 for each player $j \neq i$. Ignorance, in the strong sense of Definition 3, represents *complete* ignorance. If a player is not completely ignorant, his own strategy choice can still depend on the strategy choice of another player. Hence, even if the players are not cognizant, the assumption of strategic independence might be violated—as long as they are not completely ignorant.

4. Rational Solution

The aforementioned arguments suggest characterizing a strategic game by

$$\Gamma = \big(n, \Omega, \mathscr{F}, \mathscr{I}, A, p, u, \psi\big)$$

with $\mathscr{I} = (\mathscr{I}_1, \mathscr{I}_2, \ldots, \mathscr{I}_n)$, $A = (A_1, A_2, \ldots, A_n)$, $p = (p_1, p_2, \ldots, p_n)$, $u = (u_1, u_2, \ldots, u_n)$, and $\psi = (\psi_1, \psi_2, \ldots, \psi_n)$. This leads to the following general definition of rationality:

Definition 5 (Rationality)**.** *Consider a strategic game Γ and let $s_i^* \in \mathscr{S}_i$ be the chosen strategy of Player i. He is said to be* rational *if and only if*

$$\mathrm{E}_i\Big(u_i\big(\psi_i(s_i^*)\big)\Big) \geq \mathrm{E}_i\Big(u_i\big(\psi_i(s_i)\big)\Big), \qquad \forall\, s_i \in \mathscr{S}_i\,.$$

Table 2: Rendezvous

At the breakfast table, Mary and Joe decide to go to a restaurant after work. They consider Luigi's Trattoria and Harry's Sports Bar. Mary prefers Luigi's Trattoria, whereas Joe favors Harry's Sports Bar. However, both find it better to spend time together. Joe is in a hurry and asks Mary to reserve a table. After he has left the room, she notices that they did not agree on a restaurant and now they have absolutely no possibility of communicating with each other.

Put another way, a player is rational if and only if he chooses an optimal strategy, based on his conjectures about (his own strategy and) the strategy of each other player. In the case in which the players are cognizant, it holds that $\psi_i(s_i) = (s_i, s^i)$ for all $s_i \in \mathscr{S}_i$ and $i = 1,2,\ldots,n$. Hence, we could write "$\mathrm{E}_i\big(u_i(s_i, s^i)\big)$" instead of "$\mathrm{E}_i\big(u_i(\psi_i(s_i))\big)$." However, this would be highly misleading. The problem is that Player i does not consider s^i fixed but an implicit function of s_i. This cannot be expressed by the former notation, which suggests that s_i has no influence on s^i.

Now, the procedure of rational choice in Γ goes like this:

(i) Player i considers his available set $\{\psi_i(s_i) : s_i \in \mathscr{S}_i\}$ of conjectures.

(ii) He maximizes his subjective expected utility by choosing an optimal strategy $s_i^* \in \mathscr{S}_i$.

(iii) Nature chooses the state of the world, i.e., $\omega \in \Omega$.

(iv) Player i is informed about $I_i \in \mathscr{I}_i$ that is such that $\omega \in I_i$.

(v) Now, based on his private information, Player i performs the action $a_i = s_i^*(\omega) \in A_i$.[18]

This leads us to our definition of rational solution:

Definition 6 (Rational solution)**.** *The solution $s^* = (s_1^*, s_2^*, \ldots, s_n^*) \in \mathscr{S}$ of Γ is said to be* rational *if and only if*

$$\mathrm{E}_i\Big(u_i\big(\psi_i(s_i^*)\big)\Big) \geq \mathrm{E}_i\Big(u_i\big(\psi_i(s_i)\big)\Big), \qquad \forall\, s_i \in \mathscr{S}_i,\ i = 1,2,\ldots,n.$$

The set of rational solutions of Γ is denoted by $\mathscr{R} \subseteq \mathscr{S}$.

Hence, the solution of Γ is rational if and only if all players are rational, i.e., choose an optimal strategy. A rational solution need *not* be a correlated equilibrium. In particular, this holds true if the players are cognizant,[19] which is implicitly assumed by Aumann (1987). This contradicts Aumann's general hypothesis that correlated equilibrium is an unavoidable consequence of Bayesian rationality in strategic games (Aumann, 1987, p. 2). Since every Nash equilibrium is a correlated equilibrium, we can draw the same conclusion for Nash equilibrium.

Gul (1998) provides two alternative interpretations of Aumann's model of strategic conflict: (i) A dynamic interpretation, where the game starts at a prior stage and subsequently moves to a posterior stage, and (ii) an epistemic interpretation, which refers to the belief hierarchies of the players. He comes to the same conclusion, namely that correlated equilibrium is not a necessary condition for a (Bayes) rational solution. Levi (1997b) notes that Aumann makes the implicit

[18] Since s_i^* is constant on each element of $\mathscr{I}_i$, Player i need not know the state of the world ω.

[19] For example, consider the rational solution of the prisoners' dilemma with lawyer.

assumption that the decision of a Bayes-rational player is optimal *ex post.* He argues that this might be satisfied in games against nature, but in general it is violated in strategic games, where the decisions of the players may depend on each other. In this case, their decisions should be optimal ex ante. Thus, we may conclude that correlated equilibrium represents an artifact rather than a natural consequence of Bayesian rationality in strategic games.

The given solution concept can be demonstrated by means of the rendezvous game in Table 2. I assume for the sake of simplicity that the conjectures given by $\Psi_M(s)$ and $\Psi_J(s)$ are constant on Ω, for every strategy tuple $s \in \mathscr{S}$, where "M" indicates Mary and "J" stands for Joe. The utility functions of Mary and Joe can be obtained from the following payoff matrix:

	Joe	
Mary	Trattoria	Bar
Trattoria	$(3,2)$	$(1,1)$
Bar	$(0,0)$	$(2,3)$

We can choose $\{1,2\}$ for the action set of Mary and Joe, where "1" denotes Luigi's Trattoria and "2" stands for Harry's Sports Bar. Although the structure of the game is kept as simple as possible, we cannot derive a rational solution without specifying the conjecture functions of Mary and Joe. For example, Mary could believe that Joe is a smart guy, which means that he will call a restaurant to ask whether she has made a reservation.[20] In this case, we have that $\psi_{MJ}(1) = 1$ and $\psi_{MJ}(2) = 2$, where $\psi_{MJ}(a)$ denotes Mary's conjecture about Joe's action given that she chooses the action $a \in \{1,2\}$. Hence, irrespective of her particular choice, she believes that he will come to the right place. Mary prefers Luigi's Trattoria and so she reserves a table there. Nonetheless, she is ignorant of his actual decision. This means that her conjecture does not change if he decides to go to the wrong restaurant. More precisely, we have that

$$\Psi_M(1,1) = (1,1), \quad \Psi_M(1,2) = (1,1), \quad \Psi_M(2,1) = (2,2), \quad \Psi_M(2,2) = (2,2).$$

Here, the first argument of Ψ_M denotes Mary's action, whereas the second one is Joe's action.

Now, assume that Joe actually *calls* a restaurant. We suppose that her reservation coincides with her actual choice. Then Joe is a cognizant player, which means that

$$\Psi_J(1,1) = (1,1), \quad \Psi_J(1,2) = (1,2), \quad \Psi_J(2,1) = (2,1), \quad \Psi_J(2,2) = (2,2).$$

Here, once again, the first argument of Ψ_J is Mary's and the second one is Joe's action. Since Joe is cognizant, he knows that Mary chooses Luigi's Trattoria—irrespective of his own choice. This means that Joe's conjecture about Mary is $\psi_{JM}(a) = 1$ for $a = 1,2$ and thus his optimal choice is Luigi's Trattoria. Hence, Mary and Joe meet in the Trattoria.

[20] Joe need not call both restaurants. He will simply go to the second one if the first one answers in the negative.

5. Rational Solutions of the Prisoners' Dilemma

Γ contains our assumptions about the structure of the game, $n, \Omega, \mathscr{F}, \mathscr{I}, A$, the preferences of the players, p, u, and their conjectures about each other, i.e., ψ. Hence, in our framework, the game is much more than "simply the totality of rules which describe it" (von Neumann and Morgenstern, 1953, p. 49). In fact, the set of rational solutions of Γ depends essentially on the conjectures of the players:

(a) Suppose that there is no lawyer in the prisoners' dilemma but, nonetheless, the prisoners *believe* that the other is cognizant and plays tit for tat. Then we obtain the cooperative solution, although the players in fact are ignorant and so there is no binding agreement.

(b) If there is a lawyer but each prisoner still decides to confess, irrespective of whatever the other is going to do, the noncooperative solution evolves. In fact, this solution is rational but not Pareto efficient, although it is possible to make a binding agreement.[21]

This means that, depending on the given conjecture functions $\psi_1, \psi_2, \ldots, \psi_n$, we can justify both the cooperative and noncooperative solution of the prisoners' dilemma. In (a) it does not matter whether the prisoners *de facto* play tit for tat. The solution depends only on what the prisoners *believe* about each other. Nonetheless, the situation described in (a) might appear implausible to the reader: Why should a prisoner be confident that the other denies if (and only if) he denies? An uncomfortable feeling might occur also in (b): There is a lawyer and so the prisoners know what the other is going to do. Moreover, each prisoner knows that nobody is able to betray the other. Hence, why should the prisoners refuse to cooperate?

In the following, I present plausible solutions of the prisoners' dilemma by distinguishing between cognizant and ignorant players. It turns out that cooperation is a plausible solution if we follow Aumann's model of strategic conflict, i.e., if we assume that the players are cognizant.[22] By contrast, Aumann's model is not applicable if the players are ignorant, although this is a typical assumption when discussing the prisoners' dilemma. In this case, the noncooperative solution is plausible but, however, it cannot be explained by Aumann's model.

The prisoners' dilemma has been chosen only for illustrative purposes. After studying the rational solutions of the prisoners' dilemma, the reader should hopefully be able to understand how the given results can be translated to any n-person game. We will see that correlated equilibrium represents an exceptional case if the players are cognizant, i.e., if we follow Aumann's model of strategic conflict. Correlated equilibrium can be justified if the players are ignorant, but then we need additional assumptions, which turn out to be quite restrictive—and, at least in my opinion, are not satisfied in most real-life situations. In any case, correlated equilibrium is neither necessary nor sufficient for Bayesian rationality in strategic games.

[21] A solution $s^* \in \mathscr{S}$ is said to be Pareto efficient if and only if there is no $s \in \mathscr{S}$ such that $\mathrm{E}_i\big(u_i(s)\big) > \mathrm{E}_i\big(u_i(s^*)\big)$ and $\mathrm{E}_j\big(u_j(s)\big) \geq \mathrm{E}_j\big(u_j(s^*)\big)$ for all $j \neq i$.

[22] A similar approach can be found in Levi (1997b), who does *not* distinguish between states and actions. Thus, his arguments are probabilistic rather than epistemic but, nevertheless, his overall conclusions are quite similar.

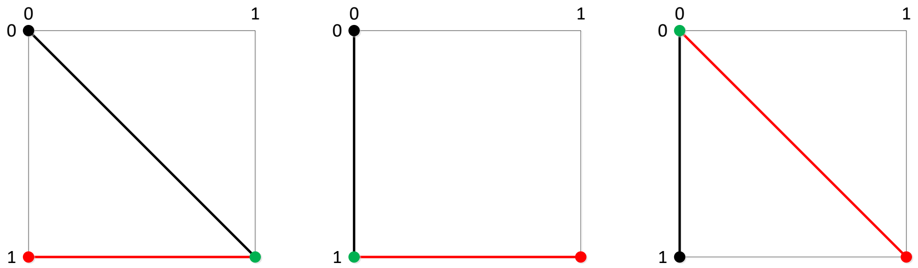

Figure 5: Response functions in the prisoners' dilemma.

5.1. Cognizant Players

In this section, we follow Aumann (1987) and assume that the players are cognizant. We have that $\mathrm{E}_i\big(u_i(s)\big) = \mathrm{E}_i\big(u_i(t)\big)$ for all $s, t \in \mathscr{R}$ and $i = 1, 2, \ldots, n$.[23] Hence, whenever the players are cognizant, the set of rational solutions, $\mathscr{R}$, is *essentially* unique and thus we may refer to "the" rational solution of Γ. Since we assume that the players are cognizant, $\psi_{ij}(s_i) = s_j$ represents the *response* of Player j given that Player i applies the strategy s_i. Thus, in this section, ψ_{ij} is referred to as the response function of Player i with respect to Player j. In a 2-person game we can simply say that ψ_{12} and ψ_{21} are the response functions of Player 1 and Player 2, respectively.

The response functions specify how the players behave and, equivalently, they are *determined* by the behavior of the players. This can be demonstrated by the prisoners' dilemma. Suppose that the response functions ψ_{12} and ψ_{21} are given by the first diagram of Figure 5.[24] The black line represents ψ_{12}, i.e., the responses of Player 2 to Player 1. Analogously, the red line carries ψ_{21}, i.e., the responses of Player 1 to Player 2. The given response functions indicate that Player 1 decides to confess, irrespective of whatever Player 2 is going to do, whereas Player 2 plays tit for tat. Hence, we obtain the noncooperative solution, which is illustrated by the green bullet point, i.e., the intersection of the two lines. Now, suppose that Player 2 decides to deny. In this case, we have to move the black line to the left. This is shown in the second diagram, in which case Player 1 confesses and Player 2 denies. If Player 1 decides to deny, we have to move up the red line, which leads to the third diagram. In this case, the solution turns out to be cooperative, etc.

Whenever we consider a strategic game Γ with cognizant players, we can a priori exclude some constellations $\psi = (\psi_1, \psi_2, \ldots, \psi_n)$. More precisely, some constellations can be discarded if we assume that the players are rational, both from an individual *and* a collective point of view:

(i) Some ψ are inconsistent, i.e., Γ has no solution at all.

(ii) Other ψ imply that $\mathscr{R} = \emptyset$, i.e., Γ has no rational solution.

(iii) Finally, ψ can lead to a rational but Pareto-inefficient solution.

Hence, if the players know each other's strategy, we may accept only those constellations of $\psi_1, \psi_2, \ldots, \psi_n$ that lead to a rational and Pareto-efficient solution. More precisely, there must be

[23] Suppose that the rational solutions $s, t \in \mathscr{R}$ are such that $\mathrm{E}_i\big(u_i(s)\big) \neq \mathrm{E}_i\big(u_i(t)\big)$ for some player $i \in \{1, 2, \ldots, n\}$. Then we have that $\mathrm{E}_i\big(u_i(s_i, s^i)\big) < \mathrm{E}_i\big(u_i(t_i, t^i)\big)$ or $\mathrm{E}_i\big(u_i(s_i, s^i)\big) > \mathrm{E}_i\big(u_i(t_i, t^i)\big)$. Hence, Player i can improve his expected utility by moving from s_i to t_i or from t_i to s_i and thus either s or t cannot be a rational solution.

[24] The responses are represented by the bullet points at the end of each line.

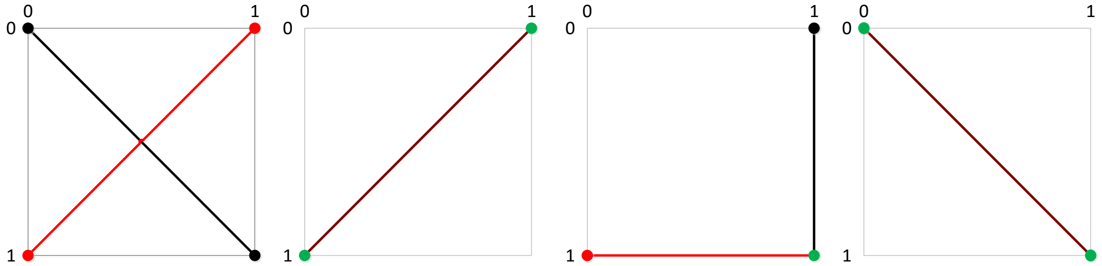

Figure 6: Prisoners' dilemma without solution (1^{st}), without rational solution (2^{nd}), with rational but Pareto-inefficient solution (3^{rd}), and with Pareto-efficient rational solution (4^{th}).

no other ψ such that the *rational* solution of Γ is better for one but not worse for another player. This might essentially restrict the set of possible candidates for ψ, which will be demonstrated, once again, by means of the prisoners' dilemma with lawyer. The response functions ψ_{12} and ψ_{21} depicted in the first diagram of Figure 6 are inconsistent. This can be seen as follows: If Player 1 chooses Action 0, he knows that Player 2 chooses Action 0, too. In this case, Player 2 knows that Player 1 chooses Action 1, which is a contradiction. In the same way, we obtain a contradiction with Action 1, and the same arguments apply, mutatis mutandis, for Player 2.[25] That is, the game has no solution at all. The second diagram represents a situation in which ψ is consistent. There are two possible solutions, which are given by the green bullet points. However, there is no rational solution. For example, if Player 1 chooses Action 1, Player 2 responds with Action 0, but, according to ψ_{21}, it would be better for Player 2 to choose Action 1, etc. Moreover, in the third diagram we obtain the noncooperative solution of the prisoners' dilemma, which is rational under the given response functions ψ_{12} and ψ_{21}. However, it is Pareto inefficient. Finally, in the fourth diagram we can find a constellation of ψ_{12} and ψ_{21} that turns cooperation into a rational and Pareto-efficient solution of the prisoners' dilemma. In this case, the prisoners play tit for tat. Although the noncooperative solution is a correlated equilibrium, it is *not* a rational solution under the given response functions.

The previous arguments reveal that the prisoners cooperate if they are cognizant and thus follow Aumann's model of strategic conflict. Here, cooperation is just a result of cognizance. The reason the players are cognizant of each other's strategy is only of secondary importance. We have already seen that cooperation takes place even if the prisoners are ignorant—provided both players are confident, i.e., believe that the other plays tit for tat. More generally, in any n-person game, a cognizant player *knows* that the others cooperate, whereas a confident player just *believes* that they do. Indeed, we could always justify a cooperative solution, which typically occurs if the players are cognizant, by making the weaker assumption that they are confident rather than cognizant.[26] However, in my opinion, it makes no sense for a player to think that the others cooperate if he assumes that they are rational and know (or, at least, *believe*) that he is not aware of their strategies.

[25] As already mentioned, we assume for the sake of simplicity that the players can choose only between "pure strategies."
[26] Once again, I am very grateful to Steve Brams for discussing this point with me in a personal communication.

5.2. Ignorant Players

Aumann's model of strategic conflict cannot be applied if the players are ignorant. If a rational player is ignorant, his conclusions are solely based on belief rather than knowledge. Thus, his strategy choice does not depend on the strategy choice of another player, and if all players are rational but ignorant, their strategy choices are mutually independent. Hence, the assumption of strategic independence, which is fundamental in noncooperative game theory, can readily be justified if the players are ignorant but not if they are cognizant. Once again, it is worth emphasizing that strategic independence has nothing to do with stochastic independence. We can imagine situations in which the strategies are stochastically independent but strategically dependent or stochastically dependent but strategically independent.[27]

In many strategic conflicts, at least, we can justify the assumption that Player i *believes* that his strategy has no influence on the others' strategies:[28]

(i) If Player i is rational, he could believe that the other players are rational, too.

(ii) Moreover, if Player i cannot observe what the others are going to do, he could think that his opponents are ignorant of his own strategy as well.[29]

Then we have that $\Psi_i(s_i, s^i) = (s_i, \bar{s}^i)$ for all s_i, where $\bar{s}^i$ contains the *conjectures* of Player i about the strategies of the other players. Since $\bar{s}^i$ is invariant under a change of s_i, we obtain the expected utility $\mathrm{E}_i(u_i(s_i, \bar{s}^i))$ for all s_i and $i = 1, 2, \ldots, n$. This means that Player i considers the other strategies *fixed* when maximizing his expected utility. He can do this precisely because he believes that his choice has absolutely no influence on the choice of any other. In fact, this makes sense only if Player i thinks that his opponents are rational but *not* cognizant. Hence, to justify the assumption of strategic independence, we must accept that our usual interpretation of Aumann's model, namely that players are cognizant, is wrong.

Now, a rational solution, $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$, is characterized by

$$\mathrm{E}_i\Big(u_i\big(s_i^*, \bar{s}^i\big)\Big) \geq \mathrm{E}_i\Big(u_i\big(s_i, \bar{s}^i\big)\Big), \qquad \forall\, s_i \in \mathscr{S}_i,\ i = 1, 2, \ldots, n.$$

Additionally, if the conjectures of all players are correct, i.e., $\bar{s}^i = s^{*i}$ for $i = 1, 2, \ldots, n$, we obtain

$$\mathrm{E}_i\Big(u_i\big(s_i^*, s^{*i}\big)\Big) \geq \mathrm{E}_i\Big(u_i\big(s_i, s^{*i}\big)\Big), \qquad \forall\, s_i \in \mathscr{S}_i,\ i = 1, 2, \ldots, n.$$

This solution is said to be a *subjective* correlated equilibrium (Aumann, 1987, p. 7). Moreover, if we assume that the players have a common prior, i.e., $p_1 = p_2 = \ldots = p_n$, we can drop the index "i" from "E_i" and obtain a correlated equilibrium (Aumann, 1987, p. 4):

$$\mathrm{E}\Big(u_i\big(s_i^*, s^{*i}\big)\Big) \geq \mathrm{E}\Big(u_i\big(s_i, s^{*i}\big)\Big), \qquad \forall\, s_i \in \mathscr{S}_i,\ i = 1, 2, \ldots, n.$$

To sum up, we have made the following assumptions to justify correlated equilibrium:

[27] The former is reflected by the prisoners' dilemma with lawyer, whereas the latter is described in Aumann (1974, 1987).

[28] This is an implicit assumption of epistemic game theory.

[29] Rendezvous is a counterexample: Even though Mary is ignorant, she believes that Joe is cognizant.

(a) The players are rational.

(b) They think that each other player is rational, too.

(c) They believe that each other player is ignorant.

(d) Their conjectures about each other's strategy are correct.

(e) They have a common prior.

It seems somewhat bizarre to assume that the players believe that the others are ignorant (c), whereas their own conjectures about the others are correct (d), but we can find some situations in which these assumptions are satisfied. For example, consider the prisoners' dilemma without lawyer and suppose that the prisoners anticipate that their fellow inmate confesses. Interestingly, (b), (c) and (d) are not necessary for correlated equilibrium: If Player 1 does *not* believe that Player 2 is rational, i.e., Player 1 expects that Player 2 is going to deny, he will confess. Now, suppose that Player 2 is rational—contrary to the expectations of Player 1. In this case, Player 2 also confesses and so we obtain a correlated equilibrium although (b) and (d) are violated. Moreover, in the rendezvous game, Mary does not believe that Joe is ignorant, which means that (c) is violated. However, meeting in the Trattoria is a correlated equilibrium.

Epistemic game theory aims at explaining the psychological conditions under which (d) is satisfied. For example, if the belief hierarchies of the players are simple and there is common belief in rationality, we obtain a Nash equilibrium where the conjectures of the players are correct (Perea, 2012, p. 149).[30] These conditions appear to be quite restrictive. In fact, (d) goes beyond our subjectivistic approach to rational choice, where the conjectures of a rational player may be incorrect—as long as he is convinced about his opinion. Hence, (d) is not necessary if we want to explain the solution of a strategic conflict and I think that this assumption can hardly be met in real-life situations since, typically, players suffer from incomplete information.

The common-prior assumption (e) goes back to Harsanyi (1967–1968). Aumann (1987, p. 12) points out that "Common priors are explicit or implicit in the vast majority of the differential information literature in economics and game theory." In fact, the common-prior assumption leads to a substantial simplification of Aumann's model of strategic conflict. Without a common prior, statements like "Strategy s_1 is stochastically independent of Strategy s_2" and "Player 1 chooses Action a_1 with probability $\frac{1}{2}$" are meaningless—unless we specify the underlying probability measure, i.e., the prior, of the corresponding player. By contrast, if the priors are distinct, it is impossible to characterize the solution of the game by a single profile distribution, i.e., a joint probability distribution of strategies. For a broad overview of the common-prior assumption in economics see Morris (1995).

Despite its wide acceptance in economics, the common-prior assumption is the subject of controversy (Aumann, 1998, Gul, 1998, Morris, 1995). Although Aumann generally supports the common-prior assumption, in Aumann (1987, p. 12) he mentions that it "is not a tautological

[30] Common belief in rationality is stronger than (a) and (b). The given argument requires also (c) and (e). Moreover, it is implicitly assumed that the strategies are stochastically independent.

consequence of the Bayesian approach." Gul (1998) goes even further and states that "the assumption of common priors in this context is antithetical to the Savage-established foundations of statistics (i.e., the 'Bayesian view'), since it amounts to asserting that at some moment in time everyone must have identical beliefs." In fact, Savage (1954, p. 3) points out that his personalistic interpretation of probability does "not deny the possibility that two reasonable individuals faced with the same evidence may have different degrees of confidence in the truth of the same proposition." Morris (1995) concludes that it makes little sense, on the one hand, to allow for subjective probabilities and, on the other hand, to impose the common-prior assumption, which postulates that the players start with the same probability beliefs before calculating their posteriors. Throughout this work, we have not made use of the common-prior assumption.

6. Conclusion

Aumann's model of strategic conflict is based on the assumption that strategic uncertainty can be translated into imperfect information and thus treated like uncertainty in a game against nature. Although the players may not know each other's action, Aumann's formal approach suggests, at least, that they know each other's strategy. Put another way, they are cognizant of whatever strategy choices the others make. Moreover, Aumann assumes that the players are Bayes rational, have a common prior, and choose their strategies independently. Under these circumstances, Aumann comes to the conclusion that correlated equilibrium is an unavoidable consequence of Bayesian rationality in strategic games.

It has been shown that cognizance and strategic independence contradict each other. If we assume that the players know each other's strategy, they choose their strategies in a coherent way. That is, a fundamental assumption of noncooperative game theory, i.e., strategic independence, is violated. Cognizance leads to cooperation and, in general, the rational solution of a strategic game is not a correlated equilibrium. By contrast, strategic independence can readily be justified if the players completely ignore each other's strategy, i.e., if their decisions are based on belief rather than knowledge. Aumann's model does not succeed in distinguishing between knowledge and belief. This problem has been resolved by introducing conjecture functions.

We are able to explain the epistemic conditions under which the solution of a strategic conflict is a correlated equilibrium. Our conclusion is that correlated equilibrium is neither necessary nor sufficient for Bayesian rationality in strategic games. This has been demonstrated by means of the prisoners' dilemma. We can make additional assumptions that guarantee that the solution of a strategic game is a correlated equilibrium. However, these assumptions turn out to be quite restrictive. Hence, in our subjectivistic framework, the overall concept of equilibrium fades into the background.

A. Appendix

This work builds on the foundations of subjectivistic decision theory, i.e., of rational choice under uncertainty. Hence, it seems worth recapitulating the basic theory for those who are not familiar with the subject matter. This appendix presents typical assumptions about (i) the structure of the decision problem, (ii) the consistency of the decision-maker's preferences, and (iii) his behavior or—in the context of game theory—the behavior of the players. A nice overview of subjectivistic expected-utility theories can be found in Fishburn (1981).

A.1. Games against Nature

The following exposition is based on Fishburn (1981) and Savage (1954). The state space of the decision problem is denoted by Ω. It is assumed that Ω is nonempty. Each element $\omega \in \Omega$ represents a state of nature or state of the world. Let $\mathscr{F} = 2^{\Omega}$ be the power set of Ω. Hence, $\mathscr{F}$ is a σ-algebra on Ω, i.e., $\Omega \in \mathscr{F}$, $F \in \mathscr{F} \Rightarrow \Omega \setminus F \in \mathscr{F}$, and $F_1, F_2, \ldots \in \mathscr{F} \Rightarrow \bigcup_{i \in \mathbb{N}} F_i \in \mathscr{F}$. Each element of $\mathscr{F}$ is referred to as an event. The event $F \in \mathscr{F}$ is said to happen if and only if some state $\omega \in F$ obtains. The tuple $(\Omega, \mathscr{F})$ represents a measurable space. A function $p \colon \mathscr{F} \to [0,1]$ is said to be a probability measure if and only if $p(\Omega) = 1$ and $p\big(\bigcup_{i \in \mathbb{N}} F_i\big) = \sum_{i \in \mathbb{N}} p(F_i)$ for all mutually disjoint events $F_1, F_2, \ldots \in \mathscr{F}$. It is supposed that the decision-maker has a subjective probability measure p, i.e., a prior, so that $p(F)$ represents his prior probability of $F \in \mathscr{F}$. This leads to a probability space $(\Omega, \mathscr{F}, p)$. Moreover, there exists a nonempty set $\mathscr{N}$ of null events, which consists of all $F \in \mathscr{F}$ with $p(F) = 0$. The decision-maker does not believe that any event $F \in \mathscr{N}$ happens and thus he may ignore every null event. The null events are called negligible, whereas each event $F \in \mathscr{F} \setminus \mathscr{N}$ is said to be substantial.

Fishburn (1981, p. 141) notes that "states ... lead to specific consequences that depend on the course of action adopted by the individual." In addition, he writes that "... the occurrence of one consequence precludes the occurrence of any other consequence." Thus, consider a nonempty set C of consequences and let $\mathscr{C}$ be the set of all functions from Ω to C. Throughout this work, two functions $s, t \in \mathscr{C}$ are considered identical if and only if they coincide for all $\omega \in \Omega$. Thus, for $s = t$ it is not sufficient that s and t are almost surely equal with respect to the subjective probability measure p. Let A with $|A| > 1$ be any set of actions. It is assumed that the state space, Ω, is properly specified. This means that if the decision-maker chooses a certain action, $a \in A$, he creates one and only one function $s \in \mathscr{C}$, whereas each state of nature, ω, leads to a specific consequence $s(\omega) \in C$. More precisely, there exists a function $f \colon A \to \mathscr{C}$ that maps each action $a \in A$ to a function $s = f(a)$ from Ω to C. Thus, $s(\omega) \equiv f(a)(\omega) \in C$ denotes the consequence of Action $a \in A$ given that the state $\omega \in \Omega$ obtains. According to Savage (1954, Chapter 2.5), "If two different acts had the same consequences in every state of the world, there would ... be no point in considering them two different acts at all." Thus, in a game against nature, it is not necessary to distinguish between the action, a, of the decision-maker and the associated element $f(a)$ of $\mathscr{C}$. Hence, we may call each element of $\mathscr{C}$ a Savage act (Fishburn, 1981, p. 143,160).

It is assumed that the decision-maker has a utility function $u\colon C \to \mathbb{R}$ such that

$$\mathrm{E}\big(u(s)\big) := \int_\Omega u\big(s(\omega)\big)\, p(d\omega) < \infty$$

for all $s \in \mathscr{C}$. The Savage act s is preferred to t, i.e., $s \succ t$, if and only if $\mathrm{E}\big(u(s)\big) > \mathrm{E}\big(u(t)\big)$. Hence, the utility function u induces an asymmetric weak order $\succ$ on $\mathscr{C}$, i.e., $\succ$ is a strict preference relation such that $s \succ t \Rightarrow \neg(t \succ s)$ and $r \succ t \Rightarrow (r \succ s \vee s \succ t)$ for all $r, s, t \in \mathscr{C}$. It can easily be extended to a weak order $\succeq$ (Fishburn, 1981, p. 145). Now, let $\mathscr{S}$ be a nonempty subset of $\mathscr{C}$. This may be seen as the set of Savage acts that are available to the decision-maker. He is considered rational if and only if he chooses an optimal act $s^* \in \mathscr{S}$. More precisely, there must be no other Savage act $s \in \mathscr{S}$ such that $\mathrm{E}\big(u(s)\big) > \mathrm{E}\big(u(s^*)\big)$. This represents a behavioral assumption of rational choice—besides our given assumptions about structure and consistency.

So far, we have assumed that the prior, p, and the utility function, u, are given. That is, the preference relation $\succ$ has been obtained as a *result* of p and u. Subjectivistic decision theory usually goes the other way around (de Finetti, 1937, Fishburn, 1981, Ramsey, 1931, Savage, 1954). This means one starts with some structural assumptions about the decision problem and basic requirements regarding the consistency of a *given* preference relation. Then he shows that the given preferences can be represented by a subjective probability measure p and a utility function u (Fishburn, 1981). Hence, a rational decision-maker acts *as if* he would maximize his subjective expected utility. Put another way, we only pretend that he knows his prior, utility function, and the available Savage acts, but we do not require that the decision-maker calculates his expected utility *de facto*. This is important for understanding the particular approach to strategic conflict chosen by Aumann (1987).

Now, the procedure of rational choice in a game against nature can be illustrated this way:

(i) The decision-maker considers his available set $\mathscr{S}$ of Savage acts.

(ii) He maximizes his subjective expected utility by choosing an optimal act $s^* \in \mathscr{S}$.

(iii) Nature chooses the state of the world, i.e., $\omega \in \Omega$.

(iv) The consequence $s^*(\omega) \in C$ of his decision, s^*, takes place.

In the Bayesian framework, the decision-maker is equipped with some private information partition $\mathscr{I}$ of Ω, i.e., a set of nonempty and mutually disjoint events whose union equals Ω. It is supposed that $\mathscr{I}$ is finite only for the sake of simplicity but without loss of generality. After nature chooses the state of the world, ω, the decision-maker is informed about the event $I \in \mathscr{I}$ that is such that $\omega \in I$. Now, he replaces his prior, p, by the posterior $p(\cdot \,|\, I)$ with $p(F \,|\, I) \propto p(I \,|\, F)\, p(F)$ for all $F, I \in \mathscr{F} \setminus \mathscr{N}$. We have that $p(F \,|\, I) = 0$ whenever $F \cap I = \emptyset$ and thus, from the decision-maker's point of view, each substantial event $I \in \mathscr{I}$ leads to a new probability space in which the posterior $p(\cdot|I)$ represents his subjective probability measure.

Whenever the decision-maker knows that some event $I \in \mathscr{I}$ happens, he chooses an action $a_I \in A$, which leads to one and only one *restricted* Savage act $s_I \in \mathscr{C}_I$ (Fishburn, 1981, p. 143,160), where $\mathscr{C}_I$ denotes the set of all functions from $I \in \mathscr{I}$ to C.[31] More precisely, there exists a function $f_I : A \to \mathscr{C}_I$ that maps each action $a_I \in A$ to a function $s_I = f_I(a_I)$ from $I \in \mathscr{I}$ to C. Hence, we require that s_I is *uniquely* determined by the choice of the decision-maker on the basis of I and thus, given the action a_I, s_I must not depend on any choice, a_J, that is made on the basis of another event $J \in \mathscr{I}$. Now, $s_I(\omega) \equiv f_I(a_I)(\omega) \in C$ denotes the consequence of the decision-maker's action based on I given that the state $\omega \in I$ obtains.

By choosing an action for each event $I \in \mathscr{I}$, the decision-maker implicitly creates a Savage act $s \in \mathscr{S}$. More precisely, s is such that $s(\omega) = s_I(\omega)$ for each event $I \in \mathscr{I}$ and state $\omega \in I$. It is supposed that each Savage act $s \in \mathscr{S}$ can be constructed in this way. Moreover, by acting on the basis of his private information, the decision-maker also creates a function from Ω to A that is constant on each element of $\mathscr{I}$. This function represents the (Bayes) strategy of the decision-maker against nature, whereas the corresponding Savage act s can be viewed as nature's "response." The decision-maker considers his strategy a control variable and nature's response a state variable. This means that he can deliberately choose any strategy that can be constructed on the basis of his private information partition, $\mathscr{I}$, and action set, A, but he must accept the associated consequence, $s(\omega)$, that occurs if the state $\omega \in \Omega$ obtains. That is, the state $\omega \in \Omega$ is not an object of choice.

A decision-maker is said to be Bayes rational if and only if he chooses an optimal Savage act, given his private information, i.e., one that maximizes his conditional expected utility $\mathrm{E}\big(u(s_I)\,|\,I\big)$ for each substantial event $I \in \mathscr{I}$. More precisely, his action must lead to a restricted Savage act s_I^* such that $\mathrm{E}\big(u(s_I^*)\,|\,I\big) \geq \mathrm{E}\big(u(s_I)\,|\,I\big)$ for all Savage acts s_I that are available if some substantial event $I \in \mathscr{I}$ happens. Let $s^* \in \mathscr{S}$ be the Savage act of a Bayes-rational decision-maker and $s \in \mathscr{S}$ another Savage act. From the law of total expectation it follows that

$$\mathrm{E}\big(u(s^*)\big) = \mathrm{E}\Big(\mathrm{E}\big(u(s_I^*)\,|\,I\big)\Big) \geq \mathrm{E}\Big(\mathrm{E}\big(u(s_I)\,|\,I\big)\Big) = \mathrm{E}\big(u(s)\big).$$

We conclude that every Bayes-rational decision-maker is rational. Conversely, each rational decision-maker is Bayes rational. Otherwise, there would exist a substantial event $I \in \mathscr{I}$ where his action is suboptimal and so the decision-maker could increase his unconditional expected utility by substituting the suboptimal action with an optimal one. Hence, Bayesian rationality and rationality (in the unconditional sense) are just two sides of the same coin. For this reason, I do not distinguish between Bayesian rationality and rationality per se.

A.2. Strategic Games

The following model of strategic conflict is introduced by Aumann (1987). Consider a strategic game with $n \in \mathbb{N} \setminus \{1\}$ players and let $(\Omega, \mathscr{F})$ be the measurable space of the game. Savage's axioms require that Ω is uncountable (Fishburn, 1981, p. 161) and thus, in contrast to Aumann (1987), I do not presume that Ω is finite. Player i is equipped with a private information partition $\mathscr{I}_i$, an action set $A_i \subseteq \mathscr{A}$ with $|A_i| > 1$, where $\mathscr{A}$ is some nonempty set, a prior p_i, and a utility

[31] In the case in which the event I is negligible, the decision-maker considers his choice based on I negligible, too.

function u_i from $\mathscr{A}^n$ to $\mathbb{R}$. The n-tuple $s = (s_1, s_2, \ldots, s_n) = \big(s_i, s^i\big)$ is said to be a strategy tuple, where s_i denotes the strategy of Player i and s^i contains the strategies of the other players. The strategy of Player i, s_i, is a function from Ω to A_i that is constant on each $I_i \in \mathscr{I}_i$.

Let $\mathscr{S}_i$ be the set of all functions from Ω to A_i that are constant on each $I_i \in \mathscr{I}_i$, i.e., the set of all strategies that are available to Player i. This means that $\mathscr{S} = \times_{i=1}^{n} \mathscr{S}_i$ is the set of all strategy tuples that are possible in the game. Further, let $\mathscr{C}$ be the set of all functions from Ω to $\mathscr{A}^n$. In a game against nature, a Savage act is the argument of the utility function of the decision-maker. The same holds true in a strategic game, but there the utility of Player i depends on his own strategy *and* the strategies of the other players. That is, the Savage act of Player i is not his own strategy, but it consists of the entire strategy tuple $s \in \mathscr{S}$. Hence, $u_i(s(\omega))$ represents the utility of Player i, provided that he applies the strategy s_i, the others apply the strategies given by s^i, and the state $\omega \in \Omega$ obtains.

Player i is said to have imperfect information if and only if there exists a substantial event $I_i \in \mathscr{I}_i$ such that $0 < p_i(I_j \,|\, I_i) < 1$ for some $I_j \in \mathscr{I}_j$ with $j \neq i$. In this case, the action of Player $j \neq i$ can be stochastic from the viewpoint of Player i. Otherwise, Player i is said to have perfect information, in which case each other's action can be considered deterministic—given the private information of Player i. According to Aumann (1987), strategic uncertainty can be treated like uncertainty in a game against nature, i.e., it can be translated into imperfect information. Moreover, he assumes that the players are Bayes rational. Hence, whenever Player i receives some substantial information, i.e., if some event $I_i \in \mathscr{I}_i$ with $p_i(I_i) > 0$ happens, he chooses an optimal action $a_i^* \in A_i$. More precisely, a_i^* is such that $\mathrm{E}_i\big(u_i\big(a_i^*, s^i\big) \,|\, I_i\big) \geq \mathrm{E}_i\big(u_i\big(a_i, s^i\big) \,|\, I_i\big)$ for all $a_i \in A_i$ (Aumann, 1987, p. 7). Here, the tuple $\big(a_i, s^i\big)$ represents a *restricted* Savage act that Player i chooses on the basis of private information.[32] Thus, $a_i(\omega)$ represents not only an action, it is also considered part of the *consequence* $\big(a_i(\omega), s^i(\omega)\big)$ of the decision of Player i, a_i, that is made on the basis of $I_i \in \mathscr{I}_i$—given that $\omega \in I_i$ obtains. Moreover, $\mathrm{E}_i(\cdot \,|\, I_i)$ denotes the expectation of Player i, based on his posterior $p(\cdot \,|\, I_i)$.

Finally, Aumann (1987) presumes that, for each substantial event $I_i \in \mathscr{I}_i$ and action $a_i \in A_i$,

(i) Player i (acts as if he) knows the responses to his action a_i, which are contained in s^i, and

(ii) s^i is invariant under a change of his action a_i.

The invariance property (ii) is crucial for the derivation of correlated equilibrium. It implies that no player has an influence on any other's strategy. Hence, if Player i chooses another strategy, the other players maintain their strategies. This is typically justified by the common idea that the players choose their strategies independent of each other. This behavioral assumption is referred to as strategic independence. If Aumann's assumptions are satisfied, the solution of the game, $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$, must be such that

$$\mathrm{E}_i\Big(u_i\big(s_i^*, s^{*i}\big)\Big) \geq \mathrm{E}_i\Big(u_i\big(s_i, s^{*i}\big)\Big), \qquad \forall\, s_i \in \mathscr{S}_i,\ i = 1, 2, \ldots, n.$$[33]

[32] Following Section A.1, we could also write "a_{iI_i}" and "$s^i_{I_i}$," but this is omitted here for notational convenience.

[33] Aumann (1987, p. 4) compares "$f := \big(f^1, \ldots, f^n\big)$" with "$\big(f^{-i}, g^i\big) := \big(f^1, \ldots, f^{i-1}, g^i, f^{i+1}, \ldots, f^n\big)$." Here f^i denotes the strategy of Player i and g^i is a function of f^i, i.e., the strategy g^i must be constant on each $I_i \in \mathscr{I}_i$, too.

This solution is said to be a *subjective* correlated equilibrium (Aumann, 1987, p. 14).

Additionally, Aumann (1987, p. 7) presumes that the players have a common prior, i.e., we have that $p_i(F) = p_j(F)$ for all $F \in \mathscr{F}$ and $i, j = 1, 2, \ldots, n$. This assumption implies that the posteriors of the players can differ only through their private information—that is, their personal evidence. Otherwise, the deviations could also be due to their individual priors $p_1, p_2, \ldots, p_n$. We conclude that the solution of the game, s^*, must be such that

$$\mathrm{E}\Big(u_i\big(s_i^*, s^{*i}\big)\Big) \geq \mathrm{E}\Big(u_i\big(s_i, s^{*i}\big)\Big), \qquad \forall\, s_i \in \mathscr{S}_i,\ i = 1, 2, \ldots, n,$$

where $\mathrm{E}(\cdot)$ denotes the expectation of each player based on the common prior. This solution is said to be a correlated equilibrium (Aumann, 1987, p. 4). Moreover, if the given strategies are stochastically independent, s^* represents a Nash equilibrium (Nash, 1951).

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