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HELMUT SCHMIDT
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Faculty of Economics and Social Sciences
Department of Mathematics/Statistics

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Jadran Dobrić, Gabriel Frahm and Friedrich Schmid

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Jadran Dobrić

WGZ BANK AG
Credit Risk Control
Ludwig-Erhard-Allee 20, D-40227 Düsseldorf, Germany
E-mail: jadran.dobric@wgzbank.de

Gabriel Frahm

Helmut Schmidt University
Faculty of Economics and Social Sciences
Department of Mathematics/Statistics
Chair for Applied Stochastics and Risk Management
Holstenhofweg 85, D-22043 Hamburg, Germany
URL: www.hsu-hh.de/stochastik
Phone: +49 (0)40 6541-2791
E-mail: frahm@hsu-hh.de

Friedrich Schmid

University of Cologne
Faculty for Economics and Social Sciences
Department of Economic and Social Statistics
Albertus-Magnus-Platz, D-50923 Cologne, Germany
E-mail: schmid@wiso.uni-koeln.de

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Supervised by: Prof. Dr. Gabriel Frahm
Chair for Applied Stochastics and
Risk Management

URL: www.hsu-hh.de/stochastik

Dependence of Stock Returns in Bull and Bear Markets

JADRAN DOBRIĆ*

Credit Risk Control

WGZ BANK AG, Düsseldorf, Germany

GABRIEL FRAHM[†]

Chair for Applied Stochastics and Risk Management

Helmut Schmidt University, Hamburg, Germany

FRIEDRICH SCHMID[‡]

University of Cologne, Germany

Abstract

Despite of its many shortcomings, Pearson's rho is often used as an association measure for stock returns. A conditional version of Spearman's rho is suggested as an alternative measure of association. This approach is purely nonparametric and avoids any kind of model misspecification. We derive hypothesis tests for the conditional rank-correlation coefficients particularly arising in bull and bear markets and study their finite-sample performance by Monte Carlo simulation. Further, the daily returns on stocks contained in the German stock index DAX 30 are analyzed. The empirical study reveals significant differences in the dependence of stock returns in bull and bear markets.

Keywords: Bear market, bootstrapping, bull market, conditional Spearman's rho, copulas, Monte Carlo simulation, Pearson's rho, stock returns.

2010 MSC: Primary 62H20, Secondary 62P05.

1. INTRODUCTION

KARL Pearson's linear correlation coefficient still seems to be the most commonly used association measure for two random variables X and Y , though its many shortcomings have been often documented (see, e.g., Embrechts et al., 2002). It is well-known that Pearson's rho is strongly affected by the marginal distributions of X and Y . Further, it only quantifies the linear dependence of X and Y , but *monotone* dependence often seems to be much more relevant. Due to these reasons its moment estimate is highly sensitive to outliers.

The random variables X and Y are said to possess a strong monotone dependence if there exist two real-valued and strictly increasing functions f and g such that $|\text{Corr}(f(X), g(Y))|$ is

*E-mail: jadran.dobric@wgzbank.de. The opinions expressed in this paper are those of the authors and do not necessarily reflect views of WGZ BANK AG.

[†]Corresponding author. Phone: +49 (0)40 6541-2791, e-mail: frahm@hsu-hh.de.

[‡]E-mail: schmid@wiso.uni-koeln.de.

close to 1. It is easy to construct situations where the linear correlation coefficient of X and Y is close to 0 but even so there exist two strictly increasing transformations f and g such that $|\text{Corr}(f(X), g(Y))| = 1$. For instance, consider the random variables $X = e^Z$ and $Y = e^{\sigma Z}$ with $\sigma > 0$ and $Z \sim \mathcal{N}(0, 1)$ (McNeil et al., 2005, p. 205). Since $\text{Corr}(\log X, \log Y) = 1$, X and Y even possess a *perfect* monotone dependence, i.e., X and Y are *comonotonic* (Nelsen, 2006, p. 32). Nevertheless, $\text{Corr}(X, Y)$ is a function of σ which can take every value between 0 (as $\sigma \rightarrow \infty$) and 1 ($\sigma = 1$).

Copula theory and the concordance measures derived thereof are a convincing alternative to the linear correlation coefficient. Due to Sklar's theorem (Sklar, 1959) it is known that a joint cumulative distribution function (c.d.f.) can be decomposed into a *copula* and the marginal cumulative distribution functions. If the latter functions are continuous, the copula is unique and represents the dependence structure of the given random variables. In the following we assume that the random variables have a continuous c.d.f. and thus a meaningful concordance measure should be a function of the copula. At least it should be invariant under monotone transformations of the corresponding random variables. Examples of such measures are Spearman's rho, Kendall's tau, Gini's gamma, and Blomquist's beta. In this paper we confine ourselves to the rank-correlation coefficient or its corresponding moment estimator, i.e., Spearman's rho. For surveys on copulas and concordance measures see, e.g., Cherubini et al. (2004), Joe (1997), and Nelsen (2006).

We investigate the contemporaneous dependence of two stock returns X and Y . More precisely, we concentrate on the question whether the dependence structure is significantly different in case of a joint upswing or downswing in the market. This question has been already investigated (see, e.g., Ang and Chen, 2002, Erb et al., 1994, Fortin and Kuzmics, 2002, Hong et al., 2007, Junker and May, 2005, Patton, 2004, Silvapulle and Granger, 2001, Vaz de Melo Mendes, 2005), but we think that the statistical methods, in particular the use of Pearson's rho, is unsatisfactory. Hence, there is space for further contributions.

A bear market is present if a large number of stocks drop down. The exact meaning of a "large number" must be specified in each individual case but, nonetheless, it is clear that in a bear market many or even most *pairs* of stocks are affected simultaneously. More precisely, we are interested in a situation where two stock returns X and Y fall short of the $100p\%$ quantiles of their corresponding cumulative distribution functions. Analogously, in a bull market it usually happens that $-X$ and $-Y$ fall short of the corresponding $100q\%$ quantiles. Here p and q have to be pre-determined. The lower p -quantile of the c.d.f. (or its absolute value) of a stock return is commonly known as the *value-at-risk*, where p is the so-called *shortfall probability*. The value-at-risk is frequently used in the area of risk management and so it seems to be a natural choice in our setting.

Our approach is purely nonparametric. Contrary to Patton (2004) and Vaz de Melo Mendes (2005) we do not fit specific copulas to the data. Specifying the copula by some parametric model can lead to erroneous conclusions if the chosen model is wrong. From our point of view it is not necessary to rely on the parametric approach if the sample size is large enough. We are interested in financial data analysis and in that context it is easy to access many thousands of daily observations. By following the nonparametric approach we avoid any kind of model misspecification.

Some authors analyze the dependence structure of outliers in financial data by using the so-called *tail-dependence coefficient* (see, e.g., Fortin and Kuzmics, 2002, Junker and May, 2005). They come to the conclusion that stock (index) returns exhibit high tail-dependence in the lower tail and low tail-dependence in the upper tail (Fortin and Kuzmics, 2002). Unfortunately, tail-dependence estimation suffers from a serious bias-variance trade-off. Dobrić and Schmid (2005) as well as Frahm et al. (2005) found that estimating the tail-dependence coefficient by *nonparametric methods* can lead to huge estimation errors even if the number of observations is large. Hence, we think that the tail-dependence coefficient is not an appropriate alternative. By contrast, we develop conditional versions of Spearman's rho to assess the dependence structure of stock returns that can be observed particularly in bull and bear markets. To the best of our knowledge, the statistical literature provides only one contribution that goes in the same direction (Jaworski and Pitera, 2013), but unlike us the latter authors focus on parametric methods.

Though we focus on computational statistics and the empirical analysis of stock returns, we have to introduce some statistical theory in order to have a formal basis for our testing procedure. This is done in Section 2, where some copula theory is presented. It allows a precise formulation of the null hypotheses to be tested. The testing procedure is described in Section 3. Further, a Monte Carlo (MC) simulation is presented in Section 4, which shows that the procedure works well for sample sizes typically available in practice. In particular, it is demonstrated that the hypothesis tests keep the prescribed error probabilities of the first kind and have sufficient power to detect violations of the null hypotheses. In Section 5 we investigate the daily returns on stocks from the German stock index *DAX 30* between 1973-01-02 and 2008-11-14 and Section 6 concludes.

2. SOME COPULA THEORY

In this section we introduce some notions from copula theory (Joe, 1997, Nelsen, 2006) which are required for understanding the testing procedure to be described below. Let X and Y be two random variables with joint c.d.f. $F(x, y) = P(X \leq x, Y \leq y)$ and marginal cumulative distribution functions $G(x) = P(X \leq x)$ and $H(y) = P(Y \leq y)$ for all $x, y \in \mathbb{R}$. The quantile functions with respect to G and H are given by $G^{-1}(p) = \inf\{x : G(x) \geq p\}$ and $H^{-1}(p) = \inf\{y : H(y) \geq p\}$ for $0 < p < 1$.

Throughout this paper we assume that G and H are continuous. Therefore, according to Sklar's theorem (Sklar, 1959), there exists a unique copula $C : [0, 1]^2 \rightarrow [0, 1]$ such that

$$F(x, y) = C(G(x), H(y)), \quad \forall x, y \in \mathbb{R}.$$

The function C is the joint c.d.f. of $U = G(X)$ and $V = H(Y)$. The rank-correlation coefficient of X and Y is given by

$$\rho := \text{Corr}(U, V) = 12 \int_{[0,1]^2} uv dC(u, v) - 3 = 12 \int_0^1 \int_0^1 C(u, v) du dv - 3. \quad (1)$$

See Nelsen (2006, p. 167) for the latter representation of ρ .

For every fixed p with $0 < p < 1$ we define

$$A_L := \{(x, y) : x \leq G^{-1}(p), y \leq H^{-1}(p)\}.$$

In the following we assume that $P\{(X, Y) \in A_L\} = C(p, p) > 0$. Consider the *conditional joint c.d.f.*

$$\begin{aligned} F_L(x, y) &:= P(X \leq x, Y \leq y | (X, Y) \in A_L) = \frac{F(x \wedge G^{-1}(p), y \wedge H^{-1}(p))}{F(G^{-1}(p), H^{-1}(p))} \\ &= \frac{C[G(x \wedge G^{-1}(p)), H(y \wedge H^{-1}(p))]}{C(p, p)}, \quad \forall x, y \in \mathbb{R}. \end{aligned}$$

The corresponding conditional marginal distribution functions are given by

$$\begin{aligned} G_L(x) &:= P(X \leq x | (X, Y) \in A_L) = F_L(x, H^{-1}(p)) \\ &= \frac{C[G(x \wedge G^{-1}(p)), p]}{C(p, p)}, \quad \forall x \in \mathbb{R}, \end{aligned}$$

and $H_L(y)$ respectively. Since G_L and H_L are continuous, according to Sklar's theorem there also exists a unique copula $C_L : [0, 1]^2 \rightarrow [0, 1]$ such that

$$F_L(x, y) = C_L(G_L(x), H_L(y)), \quad \forall x, y \in \mathbb{R}.$$

Indeed, Jaworski and Pitera (2013) call

$$C_L(u, v) = F_L(G_L^{-1}(u), H_L^{-1}(v)), \quad \forall u, v \in [0, 1],$$

a *tail conditional copula*. Similarly, in Juri and Wüthrich (2002) C_L is referred to as the *extreme tail dependence copula relative to C at the level p*. We will simply call it the *lower tail-copula* and the phrase "relative to C at the level p" will be usually dropped for convenience.

Now, we can define the lower conditional rank-correlation coefficient by using the lower tail-copula, i.e.,

$$\rho_L := 12 \int_{[0,1]^2} uv dC_L(u, v) - 3 = 12 \int_0^1 \int_0^1 C_L(u, v) du dv - 3. \quad (2)$$

An analogue definition can be found for the *upper tail-copula* C_U . This is the lower tail-copula relative to the *survival copula* according to C (Nelsen, 2006, Section 2.6), i.e.,

$$\bar{C}(u, v) := u + v - 1 + C(1 - u, 1 - v), \quad \forall u, v \in [0, 1],$$

at the level q ($0 < q < 1$). The survival copula simply corresponds to the copula of $(-X, -Y)$ and thus C_U is the copula of $(-X, -Y)$ under the condition that $(-X, -Y) \in A_U$. Here the area A_U is calculated similarly to A_L just by using the quantile functions of $-X$ and $-Y$ at q rather than the quantile functions of X and Y at p . Hence, the upper conditional rank-correlation coefficient ρ_U measures the monotone dependence of two stock returns conditional on A_U . This is a situation typically arising in a bull market. In the following we will have to guarantee that $A_L \cap A_U = \emptyset$ which is equivalent to $p + q \leq 1$.

In most cases it is not possible to derive the conditional copulas C_L or C_U in closed form. Therefore ρ_L and ρ_U cannot be calculated explicitly, but MC simulation is a convenient tool

for obtaining numerical approximations of ρ_L and ρ_U with sufficient precision. We apply this method to calculate the conditional rank-correlation coefficients for the Gauss-, t_3 -, Clayton-, and Gumbel-copula (see Table 1 and Table 2). The Gauss- and t_3 -copula are given by

$$C_{\text{Gauss}}(u, v; \theta) = \Phi_{\theta}(\Phi^{-1}(u), \Phi^{-1}(v)), \quad \forall u, v \in [0, 1],$$

where

$$\Phi_{\theta}(x, y) := \int_{-\infty}^x \int_{-\infty}^y \frac{1}{2\pi\sqrt{1-\theta^2}} \cdot \exp\left(-\frac{s^2 - 2\theta st + t^2}{2(1-\theta^2)}\right) ds dt$$

as well as

$$C_{t_3}(u, v; \theta) = t_{3,\theta}(t_3^{-1}(u), t_3^{-1}(v)), \quad \forall u, v \in [0, 1],$$

with

$$t_{3,\theta}(x, y) = \int_{-\infty}^x \int_{-\infty}^y \frac{1}{2\pi\sqrt{1-\theta^2}} \cdot \left(1 + \frac{s^2 - 2\theta st + t^2}{3(1-\theta^2)}\right)^{-\frac{5}{2}} ds dt.$$

Here t_3 denotes Student's univariate t -distribution with 3 degrees of freedom and $-1 < \theta < 1$. Note that the linear correlation coefficient is symbolized by the parameter θ rather than ρ . This is because to avoid any possibility of confusion with the (unconditional) rank-correlation coefficient of C_{Gauss} or C_{t_3} . The unconditional rank-correlation coefficient of the Gauss-copula corresponds to $\rho = 6/\pi \cdot \arcsin(\theta/2)$ (Hult and Lindskog, 2002). To the best of our knowledge there exists no such closed-form expression for the t_3 -copula.

The Clayton-copula is given by

$$C_{\text{Clayton}}(u, v; \theta) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}, \quad \forall u, v \in [0, 1],$$

with $\theta \geq 0$. In the limiting case $\theta = 0$, the Clayton-copula corresponds to the *independence* or *product copula* $\Pi(u, v) := uv$ (Nelsen, 2006, p. 11).

The Gumbel-copula can be written as

$$C_{\text{Gumbel}}(u, v; \theta) = \exp\left\{-\left[(-\log u)^{\theta} + (-\log v)^{\theta}\right]^{1/\theta}\right\}, \quad \forall u, v \in [0, 1],$$

with $\theta \geq 1$. Note that for $\theta = 1$ once again the independence copula evolves. The values of θ in Table 2 are chosen such that the *unconditional* rank-correlation coefficient corresponds to $\rho = 0.3, 0.5, 0.7$. The relationship between θ and ρ can be obtained by numerical integration or MC simulation (see Joe, 1997, p. 147).

For our approximations of the conditional rank-correlation coefficients given in Table 1 and Table 2 we use $N_{\text{MC}} = 1000$ MC replications. Each replication consists of a sample from C with sample size $n = 10^6$. Both for the simulation study and the empirical study, which follow later on, we set $p = q$. Only the Clayton-copula allows for a closed-form representation of C_L : If C is a Clayton-copula, the lower tail-copula C_L is equal to C for any $0 < p < 1$ (Juri and Wüthrich, 2002). This means ρ_L corresponds to the unconditional rank-correlation coefficient ρ .

The null hypothesis we are going to test can be formalized as

$$H_0: \rho_L = \rho_U \quad \text{vs.} \quad H_1: \rho_L \neq \rho_U,$$

where some p and q with $p + q \leq 1$ are fixed. In the present framework H_1 implies that the monotone dependence of stock returns in bear markets is not the same as in bull markets.

Gauss-copula						
	$\theta = 0.25$		$\theta = 0.50$		$\theta = 0.75$	
$p = q$	lower	upper	lower	upper	lower	upper
0.05	.0404 (.0004)	.0407 (.0004)	.1109 (.0003)	.1114 (.0003)	.2622 (.0002)	.2624 (.0002)
0.20	.0601 (.0001)	.0601 (.0001)	.1595 (.0001)	.1593 (.0001)	.3485 (.0001)	.3483 (.0001)
0.35	.0775 (.0001)	.0774 (.0001)	.1972 (.0001)	.1973 (.0001)	.4090 (.0001)	.4091 (.0001)
0.50	.0962 (.0001)	.0962 (.0001)	.2354 (.0001)	.2356 (.0001)	.4655 (.0000)	.4656 (.0000)

t_3 -copula						
	$\theta = 0.25$		$\theta = 0.50$		$\theta = 0.75$	
$p = q$	lower	upper	lower	upper	lower	upper
0.05	.3373 (.0003)	.3369 (.0003)	.4043 (.0002)	.4044 (.0002)	.5264 (.0002)	.5265 (.0002)
0.20	.3186 (.0001)	.3183 (.0001)	.3968 (.0001)	.3967 (.0001)	.5361 (.0001)	.5361 (.0001)
0.35	.2984 (.0001)	.2984 (.0001)	.3913 (.0001)	.3913 (.0001)	.5484 (.0001)	.5485 (.0001)
0.50	.2756 (.0001)	.2756 (.0001)	.3882 (.0001)	.3882 (.0001)	.5651 (.0000)	.5652 (.0000)

Table 1: MC approximations of ρ_L and ρ_U for the Gauss- and t_3 -copula possessing different values of θ . We use $N_{MC} = 1000$ MC replications, each one generating a sample from the corresponding copula with sample size $n = 10^6$. The standard errors of the approximations are given in parentheses.

Instead of a two-sided hypothesis test, a one-sided test like

$$H_0: \rho_L \leq \rho_U \quad \text{vs.} \quad H_1: \rho_L > \rho_U$$

is of general interest, since H_1 implies that the monotone dependence is higher in bear markets than in bull markets.

The null hypothesis $H_0: \rho_L = \rho_U$ stated above might be also of interest in another context. Both in theory and application of copulas it is sometimes questionable whether the random vector (X, Y) is *radially symmetric* or not (Nelsen, 2006, Section 2.7). Radial symmetry is a useful property which implies that $\rho_L = \rho_U$ since, in case (X, Y) is radially symmetric, C and the corresponding survival copula \bar{C} coincide. Hence, in order to test the null hypothesis H'_0 : “The random vector (X, Y) is radially symmetric,” one can apply the two-sided test mentioned above and reject H'_0 if H_0 is rejected.

3. THE TESTING PROCEDURE

In this section we describe the testing procedure. The first part requires independent and identically distributed (i.i.d.) data. It is well-known that short-term stock returns typically exhibit strong patterns of serial dependence. However, the i.i.d. assumption may serve as an appropriate starting point and there might exist several applications beyond financial data analysis where this assumption is adequate. Afterwards we will drop the i.i.d. assumption and explain how the test can be modified to account for the purpose of time series analysis.

Clayton-copula						
	$\theta = 0.51$		$\theta = 1.08$		$\theta = 2.13$	
$p = q$	lower	upper	lower	upper	lower	upper
0.05	.3004 (.0002)	.0025 (.0005)	.5001 (.0002)	.0018 (.0004)	.7002 (.0001)	.0035 (.0004)
0.20	.3003 (.0001)	.0040 (.0001)	.4999 (.0001)	.0113 (.0001)	.7000 (.0001)	.0318 (.0001)
0.35	.3001 (.0001)	.0130 (.0001)	.4999 (.0001)	.0356 (.0001)	.7000 (.0000)	.0906 (.0001)
0.50	.3001 (.0001)	.0298 (.0001)	.5000 (.0000)	.0764 (.0001)	.7000 (.0000)	.1783 (.0001)

Gumbel-copula						
	$\theta = 1.26$		$\theta = 1.54$		$\theta = 2.07$	
$p = q$	lower	upper	lower	upper	lower	upper
0.05	.0319 (.0004)	.3499 (.0002)	.0697 (.0003)	.4504 (.0002)	.1431 (.0003)	.5849 (.0001)
0.20	.0515 (.0001)	.3158 (.0001)	.1106 (.0001)	.4392 (.0001)	.2206 (.0001)	.5871 (.0001)
0.35	.0697 (.0001)	.2906 (.0001)	.1476 (.0001)	.4314 (.0001)	.2843 (.0001)	.5916 (.0000)
0.50	.0912 (.0001)	.2744 (.0001)	.1885 (.0001)	.4276 (.0001)	.3507 (.0000)	.5990 (.0000)

Table 2: MC approximations of ρ_L and ρ_U for the Clayton- and Gumbel-copula possessing different values of θ . We use $N_{MC} = 1000$ MC replications, each one generating a sample from the corresponding copula with sample size $n = 10^6$. The standard errors of the approximations are given in parentheses.

3.1. Independent and Identically Distributed Data

Let $\{(X_1, Y_1), \dots, (X_n, Y_n)\}$ be a sample from an i.i.d. sequence $\{(X_t, Y_t)\}_{t \in \mathbb{Z}}$ of pairs of stock returns. We estimate the marginal cumulative distribution functions G and H by

$$\hat{G}_n(x) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\{X_i \leq x\}} \quad \text{and} \quad \hat{H}_n(y) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\{Y_i \leq y\}}.$$

The corresponding estimate of the quantile function G^{-1} is given by

$$\hat{G}_n^{-1}(p) = \inf\{x: \hat{G}_n(x) \geq p\}$$

and H^{-1} is estimated by $\hat{H}_n^{-1}(p)$, respectively. For every fixed p and q with $p + q \leq 1$ we can define

$$\hat{A}_L := \{(x, y): x \leq \hat{G}_n^{-1}(p), y \leq \hat{H}_n^{-1}(p)\}$$

and \hat{A}_U in the same manner. We also define the sample sizes $n_L := |\hat{A}_L|$ with respect to the lower left and $n_U := |\hat{A}_U|$ with respect to the upper right area of the empirical distribution of X and Y (here $|\cdot|$ denotes the cardinality of a set).

The observations in \hat{A}_L and \hat{A}_U can be used for estimating ρ_L and ρ_U . More precisely, let $\mathbf{I}_{\hat{A}_L}$ be the set of all indices $i \in \{1, \dots, n\}$ such that $(X_i, Y_i) \in \hat{A}_L$. Further, let $r_{L,n}(X_i)$ and $r_{L,n}(Y_i)$ be the rank numbers of X_i and Y_i with respect to all $(X_i, Y_i) \in \hat{A}_L$. Now, according to

the left-hand side of Eq. 2, we can estimate ρ_L by

$$\hat{\rho}_{L,n} = \frac{12}{n_L} \sum_{i \in I_{\hat{A}_L}} \frac{r_{L,n}(X_i)}{n_L} \cdot \frac{r_{L,n}(Y_i)}{n_L} - 3.$$

The definition of the estimator $\hat{\rho}_{U,n}$ follows correspondingly, just by using the observations in the upper right area \hat{A}_U of the empirical distribution of X and Y .

Note that the number of data points that fall into \hat{A}_L , i.e., n_L , is a *random variable*. To clarify this point, suppose for the sake of simplicity that $\hat{A}_L = A_L$. Every draw from the entire distribution is i.i.d. and so the number of draws falling into A_L is Bernoulli distributed with parameter $\pi = C(p, p) > 0$. Hence, it is clear that $n_L^{-1} = o_p(1)$ as $n \rightarrow \infty$. The same holds for A_U , i.e., $n_U^{-1} = o_p(1)$ as $n \rightarrow \infty$, provided $\bar{C}(q, q) > 0$. Moreover, since $p + q \leq 1$, A_L and A_U do not overlap and so, *conditional* on any realized pair tuple (n_L, n_U) of numbers of draws falling into A_L and A_U , the data in the lower left and upper right area of F are independent of each other. More precisely, given some *fixed* numbers n_L and n_U , every statistic based on the data in A_L is independent of any (other) statistic based on the data in A_U . It is worth emphasizing that this kind of *conditional* independence is implicitly assumed in almost every application of statistical theory, since in most practical situations the number of observations is stochastic but, nevertheless, it is treated like a real number. In the same way we will treat n_L and n_U as real numbers in the subsequent analysis.

Schmid and Schmidt (2006) have already shown that Spearman's rho is consistent and asymptotically normally distributed. Hence, the conditional versions of Spearman's rho share the same property, i.e.,

$$\sqrt{n_L} (\hat{\rho}_{L,n} - \rho_L) \xrightarrow{d} \mathcal{N}(0, \sigma_L^2) \quad \text{and} \quad \sqrt{n_U} (\hat{\rho}_{U,n} - \rho_U) \xrightarrow{d} \mathcal{N}(0, \sigma_U^2)$$

as $n_L, n_U \rightarrow \infty$, provided the lower left and upper right tail-copulas exist.

Theorem 1 *Let the joint c.d.f. of (X, Y) be continuous. Further, suppose that the partial derivatives of the corresponding copula C exist and are continuous, too. Define $\Delta\hat{\rho}_n := \hat{\rho}_{L,n} - \hat{\rho}_{U,n}$ and $\Delta\rho := \rho_L - \rho_U$ with shortfall probabilities $p, q > 0$ such that $p + q \leq 1$. If $C(p, p), \bar{C}(q, q) > 0$ then*

$$\sqrt{n} (\Delta\hat{\rho}_n - \Delta\rho) \xrightarrow{d} \mathcal{N}(0, \tau^2), \quad n \rightarrow \infty,$$

with

$$\tau^2 = \frac{\sigma_L^2}{C(p, p)} + \frac{\sigma_U^2}{\bar{C}(q, q)}.$$

Proof. Note that $n_L/n \xrightarrow{\text{a.s.}} C(p, p)$ as well as $n_U/n \xrightarrow{\text{a.s.}} \bar{C}(q, q)$ as $n \rightarrow \infty$. This means

$$\sqrt{n} (\hat{\rho}_{L,n} - \rho_L) = \sqrt{\frac{n}{n_L}} \sqrt{n_L} (\hat{\rho}_{L,n} - \rho_L) \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma_L^2}{C(p, p)}\right), \quad n \rightarrow \infty,$$

and also $\sqrt{n} (\hat{\rho}_{U,n} - \rho_U) \xrightarrow{d} \mathcal{N}(0, \sigma_U^2 / \bar{C}(q, q))$ as $n \rightarrow \infty$. Since $p + q \leq 1$, $\hat{\rho}_{L,n}$ and $\hat{\rho}_{U,n}$ are asymptotically independent. This leads immediately to the asymptotic variance τ^2 of

$$\sqrt{n} (\Delta\hat{\rho}_n - \Delta\rho) = \sqrt{n} (\hat{\rho}_{L,n} - \rho_L) - \sqrt{n} (\hat{\rho}_{U,n} - \rho_U)$$

which is given in the theorem. ■

In practical applications p and q have to be sufficiently large such that n_L and n_U do not become too small. A typical rule of thumb might be given by $n_L, n_U \geq 40$. Suppose for the moment that C corresponds to the product copula. In this case it is expected to meet p^2n observations in the lower left part and q^2n in the upper right part of the empirical copula. This means the shortfall probabilities should be such that $p, q \geq \sqrt{40/n}$. For a sample size of $n = 1000$, i.e., an observation period of approximately 4 years, p and q should not be smaller than 0.2. In general the product copula is not appropriate to describe financial data, since in most cases there is some sort of *positive* dependence between stock returns. Hence, we can expect to have even *more* observations in the corresponding corners of the empirical copula. Thus our rule of thumb guarantees that there are always enough data to make the asymptotic results applicable.

The asymptotic variances σ_L^2 and σ_U^2 depend on the tail-copulas C_L and C_U . In general they cannot be calculated explicitly (Schmid and Schmidt, 2006). The same holds for the asymptotic variance of $\sqrt{n}(\Delta\hat{\rho}_n - \Delta\rho)$, i.e., τ^2 . However, the latter can be approximated by a simple bootstrap. For conducting the hypothesis tests one has to choose an appropriate significance level $\alpha > 0$ as well as the shortfall probabilities $p > 0$ and $q > 0$ such that $p + q \leq 1$. Now the test procedure reads as follows:

1. Compute $\hat{\rho}_{L,n}$ and $\hat{\rho}_{U,n}$ from the observations in \hat{A}_L and \hat{A}_U .
2. Compute N_B bootstrap replications from the entire sample. For each replication calculate $\hat{\rho}_{L,n}$ and $\hat{\rho}_{U,n}$ as well as the corresponding difference $\Delta\hat{\rho}_n$.
3. Estimate the asymptotic variance τ^2 of $\Delta\hat{\rho}_n$ from the bootstrap and calculate the test statistic $T = \sqrt{n} \Delta\hat{\rho}_n / \hat{\tau}$, where $\hat{\tau}^2$ is the bootstrap estimate of τ^2 .

- 4a. Reject $H_0: \rho_L = \rho_U$ if

$$|T| > \Phi^{-1}\left(1 - \frac{\alpha}{2}\right),$$

where Φ denotes the standard normal c.d.f.

The one-sided hypothesis tests differ from the two-sided test only in the fourth step:

- 4b. Reject $H_0: \rho_L \leq \rho_U$ or $H_0: \rho_L \geq \rho_U$ if $T > \Phi^{-1}(1 - \alpha)$ or $T < \Phi^{-1}(\alpha)$, respectively.

3.2. Serially Dependent Data

Now, let $\{(X_1, Y_1), \dots, (X_n, Y_n)\}$ be a sample from a strictly stationary process $\{(X_t, Y_t)\}_{t \in \mathbb{Z}}$. It is assumed that this process has a weak serial dependence structure. This means the one-sided processes $\{(X_t, Y_t)\}_{t \leq 0}$ and $\{(X_t, Y_t)\}_{t \geq l}$ become “sufficiently fast” independent as $l \rightarrow \infty$, i.e., as the number of lags between the two processes grows to infinity (Politis, 2003). To put it another way, $\{(X_t, Y_t)\}_{t \in \mathbb{Z}}$ satisfies a *strong mixing condition* and it can be shown that many time series models frequently used in the literature are strong mixing (Doukhan et al., 2009).

It is important to account for serial dependence when analyzing financial data. Especially, in periods of great turbulence the lower and upper conditional Spearman’s rho might be strongly correlated and in that case the asymptotic result given in the last section is void.

However, it can be assumed that $\Delta\hat{\rho}_n$ remains asymptotically normally distributed under a weak serial dependence structure of stock returns. This means

$$\sqrt{n}(\Delta\hat{\rho}_n - \Delta\rho) \xrightarrow{d} \mathcal{N}\left(0, \tau_{\text{LR}}^2\right), \quad n \rightarrow \infty, \quad (3)$$

where τ_{LR}^2 represents a *long-run variance* and in general we have that $\tau_{\text{LR}}^2 > \tau^2$. This assumption seems natural, since the weak convergence property of $\sqrt{n}(\Delta\hat{\rho}_n - \Delta\rho)$ is based on the weak convergence of an empirical copula process (Schmid and Schmidt, 2006). By using a functional central limit theorem for stochastic processes and applying the functional delta method one can justify the weak convergence property under an appropriate strong mixing condition.

For example, suppose that the following strong mixing condition is satisfied: The tuple (U, V) is η -dependent (Doukhan et al., 2009), i.e.,

$$\sup |\text{Cov}(f(X_{s_1}, \dots, X_{s_m}), g(X_{t_1}, \dots, X_{t_n}))| \leq (m \text{Lip } f + n \text{Lip } g) \eta_l,$$

where the supremum is taken over the set of all measurable functions f and g bounded by 1 such that $\text{Lip } f, \text{Lip } g < \infty$ and over all index sets $\{s_1, \dots, s_m\}$ and $\{t_1, \dots, t_n\}$ with $m, n \in \mathbb{N}$ such that $s_1 \leq \dots \leq s_m \leq t_1 \leq \dots \leq t_n$ with fixed lag $l = t_1 - s_m \geq 0$. Here $\text{Lip } h$ denotes the Lipschitz modulus of h on the Euclidean space, i.e.

$$\text{Lip } h := \sup_{x \neq y} \frac{|h(x) - h(y)|}{\|x - y\|_1}.$$

Further, $\{\eta_l\}_{l \geq 0}$ is a sequence of positive numbers such that $\eta_l \rightarrow 0$ as $l \rightarrow \infty$. In the following it is assumed that $\eta_l = o(l^{-2-\sqrt{5}})$.

Theorem 2 (Doukhan et al. (2009)) *Suppose that the first partial derivatives of the copula C of X and Y exist and are continuous. If the aforementioned strong mixing condition is satisfied, we have that*

$$\sqrt{n}(\hat{C}_n - C) \xrightarrow{d} \mathbf{G}, \quad n \rightarrow \infty,$$

in the Skorohod space $\mathcal{D}([0, 1]^2)$ endowed with the Skorohod metric. More precisely,

$$\mathbf{G}(u, v) = \mathbb{B}(u, v) - \frac{\partial C(u, v)}{\partial u} \cdot \mathbb{B}(u, 1) - \frac{\partial C(u, v)}{\partial v} \cdot \mathbb{B}(1, v),$$

where \mathbb{B} is a Brownian bridge on $[0, 1]^2$ with covariance function

$$\text{Cov}(\mathbb{B}(u, v), \mathbb{B}(u', v')) = \sum_{t \in \mathbb{Z}} \text{Cov}\left(\mathbf{1}_{\{U_0 \leq u, V_0 \leq v\}}, \mathbf{1}_{\{U_t \leq u', V_t \leq v'\}}\right)$$

for all $(u, v), (u', v') \in [0, 1]^2$.

Now, let f be any \mathbb{R}^2 -valued function on a subset of $\mathcal{D}([0, 1]^2)$ and suppose that f is Hadamard-differentiable at C tangentially to the functional space spanned by the centered Gaussian process \mathbf{G} given in Theorem 2. From van der Vaart (1998, Theorem 20.8) it follows that

$$\sqrt{n}(f(\hat{C}_n) - f(C)) \xrightarrow{d} f'_C(\mathbf{G}), \quad n \rightarrow \infty,$$

where f'_C is the Hadamard derivative of f at C . Since f'_C is a continuous linear map and \mathbf{G} is a tight centered Gaussian process, it follows that $f'_C(\mathbf{G}) \sim \mathcal{N}_2(0, \Omega)$ (van der Vaart, 1998, p. 299).

The Hadamard derivative of any continuous and linear function is the function by itself. In particular, we can see from Eq. 2 that the conditional versions of Spearman's rho are continuous and linear in C_L and C_U , respectively. If $C \mapsto (C_L, C_U)$ is Hadamard differentiable, too, the chain rule (van der Vaart, 1998, Theorem 20.9) yields the Hadamard differentiability of $C \mapsto (\rho_L, \rho_U)$ and we obtain

$$\sqrt{n} \left(\begin{bmatrix} \hat{\rho}_{L,n} \\ \hat{\rho}_{U,n} \end{bmatrix} - \begin{bmatrix} \rho_L \\ \rho_U \end{bmatrix} \right) \xrightarrow{d} \mathcal{N}_2(0, \Omega), \quad n \rightarrow \infty.$$

Now, the continuous mapping theorem immediately leads to the desired weak convergence property of the statistic $\sqrt{n} (\Delta\hat{\rho}_n - \Delta\rho)$ expressed by (3).

There exist many possible techniques for estimating the long-run variance τ_{LR}^2 . Due to the vast computational power which is available these days, we focus on sub-sampling and block-bootstrapping (Politis, 2003). The η -mixing condition is somewhat stronger than the strong mixing condition which can be typically found in the literature related to sub-sampling and bootstrapping. More precisely, η -mixing implies α -mixing with mixing rate $\alpha_l = o(l^{-1})$. Hence, sub-sampling and block-bootstrapping can be used in our context to estimate τ_{LR}^2 .

However, sub-sampling is probably not the best choice in our setting. The reason is that for getting an unbiased estimate of τ_{LR}^2 , the number of observations within each sub-sample must be considerably small relative to the overall sample size. In our context, only a small part of each sub-sample can be used for calculating $\hat{\rho}_{L,n}$ and $\hat{\rho}_{U,n}$, but for a proper approximation of the long-run variance it has also to be guaranteed that each sub-sample contains a sufficiently large number of usable observations.

For this reason, we concentrate on a block-bootstrap procedure suggested by Künsch (1989). Consider a block size b with $0 < b < n$ and the corresponding $n - b + 1$ blocks, i.e.,

$$B_i = \{(X_{i+1}, Y_{i+1}), \dots, (X_{i+b}, Y_{i+b})\}, \quad i = 0, 1, \dots, n - b.$$

Every bootstrap replication is given by drawing $k = \lfloor n/b \rfloor$ blocks with replacement from the entire sample and concatenating these blocks to form a new pseudo-series of stock returns. The latter consists of $kb \approx n$ pseudo-observations and every bootstrap replication leads to a pseudo-realization of $\Delta\hat{\rho}_n$. This means we obtain N_B pseudo-realizations of $\Delta\hat{\rho}_n$ which can be used to approximate τ_{LR}^2 . Finally, the test statistic is given by $T = \sqrt{n} \Delta\hat{\rho}_n / \hat{\tau}_{LR}$, where $\hat{\tau}_{LR}^2$ represents the approximated long-run variance. Under the regularity conditions mentioned in Theorem 2, the estimator $\hat{\tau}_{LR}^2$ is consistent for τ_{LR}^2 as $b \rightarrow \infty$ and $n/b \rightarrow \infty$. Hence, T can be used in the same way as the test statistic discussed in Section 3.1.

4. FINITE-SAMPLE PROPERTIES

In this section we investigate the finite-sample properties of the testing procedure described in Section 3.1. The results are obtained by MC simulations for various special cases which are essentially defined by the copula under study. The testing procedure developed in Section 3.2 could have been also investigated in the same manner, but we think that this would go beyond the scope of this paper.

First we are interested in the rejection probability of the procedure if $H_0: \rho_L = \rho_U$ is true and α is the prescribed error probability of the first kind. We consider the Gauss- and t_3 -copula

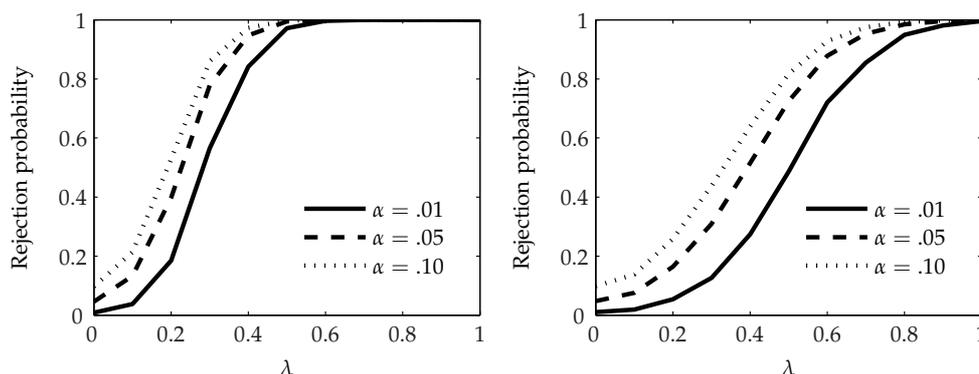


Figure 1: Power of the two-sided hypothesis test for C_{Mix1} (left-hand side) and C_{Mix2} (right-hand side) as a function of λ . The results are obtained by MC simulation for the sample size $n = 2500$, $N_B = 1000$ bootstrap replications, and $N_{\text{MC}} = 10^4$ MC replications using the shortfall probability $p = q = 0.5$.

which belong to the class of elliptical copulas. Elliptical copulas are radially symmetric which means that the aforementioned null hypothesis is true.

The selected values of the copula parameter are $\theta = 0.25, 0.5, 0.75$, the values of p are given by $p = 0.2, 0.35, 0.5$, and we validate the error probabilities $\alpha = 0.01, 0.05, 0.1$. The simulated sample size is $n = 2500$ (this means approximately 10 trading years), the number of bootstrap replications amounts to $N_B = 1000$, and the number of MC replications is $N_{\text{MC}} = 1000$. The results of the simulations are summarized in Panel 1 of Table 3. We can see that the approximated rejection probabilities satisfactorily agree with the prescribed error probabilities.

We are also interested in the *power* of the testing procedure, i.e., the probability of rejection if H_0 is wrong. For that purpose we consider the Clayton- and the Gumbel-copula. It is well-known that these copulas are not radially symmetric and thus in general $\rho_L \neq \rho_U$. Remember that the parameter θ of both copula families (see p. 5) has been selected in such a way that the unconditional rank-correlation coefficients are equal to $\rho = 0.3, 0.5, 0.7$. The results of the MC simulations are given in Panel 2 of Table 3. It can be seen that for every fixed p and α the power is an increasing function of θ . This is because the asymmetry of the Archimedean copulas C_{Clayton} and C_{Gumbel} increases with θ (see Nelsen, 2006, Ch. 4).

Similar results are obtained for the two one-sided tests which can be taken from Table 4 and Table 5. The rejection probabilities become very large whenever H_1 is true. By contrast, if H_0 is true, our simulations produce no false rejection. For instance, consider the right-sided test $H_0: \rho_L \leq \rho_U$ vs. $H_1: \rho_L > \rho_U$. In that case the null hypothesis is fulfilled for the Gumbel-copula. Panel 2 of Table 4 shows that there is no rejection for any given unconditional rank-correlation coefficient ρ , shortfall probability p , and error probability α . Given the Clayton-copula, the alternative hypothesis is true and, consequently, the rejection probabilities are very high (for example, roughly 90% for $\rho = 0.3$, $p = 0.2$, and $\alpha = 0.1$). Moreover, for $\rho = 0.5$ and $\rho = 0.7$, H_0 is rejected for the Clayton-copula in almost every simulated case.

Now we want to investigate the relationship between asymmetry and power. For that purpose we consider the mixed copula

$$C_{\text{Mix1}}(u, v; \lambda, \theta_0, \theta_1) := \lambda C_{\text{Clayton}}(u, v; \theta_1) + (1 - \lambda) C_{\text{Gauss}}(u, v; \theta_0),$$

$H_0: \rho_L = \rho_U$ vs. $H_1: \rho_L \neq \rho_U$							
Panel 1		$\theta = 0.25$		$\theta = 0.50$		$\theta = 0.75$	
$p = q$	α	Gauss	t_3	Gauss	t_3	Gauss	t_3
0.20	0.10	.091 (.0091)	.083 (.0087)	.081 (.0086)	.085 (.0088)	.091 (.0091)	.093 (.0092)
	0.05	.043 (.0064)	.047 (.0067)	.039 (.0061)	.041 (.0063)	.048 (.0068)	.048 (.0068)
	0.01	.008 (.0028)	.011 (.0033)	.006 (.0024)	.007 (.0026)	.011 (.0033)	.011 (.0033)
0.35	0.10	.106 (.0097)	.081 (.0086)	.092 (.0091)	.108 (.0098)	.095 (.0093)	.087 (.0089)
	0.05	.057 (.0073)	.038 (.0060)	.049 (.0068)	.053 (.0071)	.048 (.0068)	.051 (.0070)
	0.01	.015 (.0038)	.009 (.0030)	.011 (.0033)	.007 (.0026)	.006 (.0024)	.013 (.0036)
0.50	0.10	.104 (.0097)	.088 (.0090)	.088 (.0090)	.113 (.0100)	.089 (.0090)	.098 (.0094)
	0.05	.060 (.0075)	.043 (.0064)	.035 (.0058)	.056 (.0073)	.048 (.0068)	.049 (.0068)
	0.01	.019 (.0043)	.011 (.0033)	.008 (.0028)	.008 (.0028)	.006 (.0024)	.011 (.0033)
Panel 2		$\rho = 0.30$		$\rho = 0.50$		$\rho = 0.70$	
$p = q$	α	Clayton	Gumbel	Clayton	Gumbel	Clayton	Gumbel
0.20	0.10	.815 (.0123)	.752 (.0137)	1.000 (.0000)	.965 (.0058)	1.000 (.0000)	.999 (.0010)
	0.05	.715 (.0143)	.635 (.0152)	.999 (.0010)	.938 (.0076)	1.000 (.0000)	.996 (.0020)
	0.01	.456 (.0158)	.371 (.0153)	.993 (.0026)	.800 (.0126)	1.000 (.0000)	.999 (.0010)
0.35	0.10	.988 (.0034)	.926 (.0083)	1.000 (.0000)	.999 (.0010)	1.000 (.0000)	1.000 (.0000)
	0.05	.981 (.0043)	.876 (.0104)	1.000 (.0000)	.995 (.0022)	1.000 (.0000)	1.000 (.0000)
	0.01	.928 (.0082)	.704 (.0144)	1.000 (.0000)	.980 (.0044)	1.000 (.0000)	1.000 (.0000)
0.50	0.10	.999 (.0010)	.974 (.0050)	1.000 (.0000)	1.000 (.0000)	1.000 (.0000)	1.000 (.0000)
	0.05	.999 (.0010)	.945 (.0072)	1.000 (.0000)	1.000 (.0000)	1.000 (.0000)	1.000 (.0000)
	0.01	.996 (.0020)	.837 (.0117)	1.000 (.0000)	.995 (.0022)	1.000 (.0000)	1.000 (.0000)

Table 3: MC approximations of the rejection probabilities for the Gauss- and t_3 -copula (Panel 1) and for the Clayton- and Gumbel-copula (Panel 2) given $H_0: \rho_L = \rho_U$. The simulated sample size is $n = 2500$, the number of bootstrap replications corresponds to $N_B = 1000$, and the number of MC replications is $N_{MC} = 1000$. The standard errors for the approximated rejection probabilities are given in parentheses.

$H_0: \rho_L \leq \rho_U$ vs. $H_1: \rho_L > \rho_U$							
Panel 1		$\theta = 0.25$		$\theta = 0.50$		$\theta = 0.75$	
$p = q$	α	Gauss	t_3	Gauss	t_3	Gauss	t_3
0.20	0.10	.091 (.0091)	.096 (.0093)	.099 (.0094)	.095 (.0093)	.096 (.0093)	.099 (.0094)
	0.05	.047 (.0067)	.041 (.0063)	.040 (.0062)	.041 (.0063)	.047 (.0067)	.049 (.0068)
	0.01	.009 (.0030)	.007 (.0026)	.006 (.0024)	.007 (.0026)	.013 (.0036)	.009 (.0030)
0.35	0.10	.103 (.0096)	.093 (.0092)	.086 (.0089)	.099 (.0094)	.097 (.0094)	.094 (.0092)
	0.05	.053 (.0071)	.049 (.0068)	.038 (.0060)	.044 (.0065)	.047 (.0067)	.044 (.0065)
	0.01	.012 (.0034)	.010 (.0031)	.008 (.0028)	.006 (.0024)	.008 (.0028)	.011 (.0033)
0.50	0.10	.109 (.0099)	.092 (.0091)	.103 (.0096)	.110 (.0099)	.086 (.0860)	.094 (.0092)
	0.05	.050 (.0069)	.046 (.0066)	.046 (.0066)	.053 (.0071)	.049 (.0068)	.054 (.0071)
	0.01	.015 (.0038)	.011 (.0033)	.011 (.0033)	.011 (.0033)	.010 (.0031)	.008 (.0028)
Panel 2		$\rho = 0.30$		$\rho = 0.50$		$\rho = 0.70$	
$p = q$	α	Clayton	Gumbel	Clayton	Gumbel	Clayton	Gumbel
0.20	0.10	.899 (.0095)	.000 (.0000)	1.000 (.0000)	.000 (.0000)	1.000 (.0000)	.000 (.0000)
	0.05	.815 (.0123)	.000 (.0000)	1.000 (.0000)	.000 (.0000)	1.000 (.0000)	.000 (.0000)
	0.01	.574 (.0156)	.000 (.0000)	.997 (.0017)	.000 (.0000)	1.000 (.0000)	.000 (.0000)
0.35	0.10	.997 (.0017)	.000 (.0000)	1.000 (.0000)	.000 (.0000)	1.000 (.0000)	.000 (.0000)
	0.05	.988 (.0034)	.000 (.0000)	1.000 (.0000)	.000 (.0000)	1.000 (.0000)	.000 (.0000)
	0.01	.961 (.0061)	.000 (.0000)	1.000 (.0000)	.000 (.0000)	1.000 (.0000)	.000 (.0000)
0.50	0.10	1.000 (.0000)	.000 (.0000)	1.000 (.0000)	.000 (.0000)	1.000 (.0000)	.000 (.0000)
	0.05	.999 (.0010)	.000 (.0000)	1.000 (.0000)	.000 (.0000)	1.000 (.0000)	.000 (.0000)
	0.01	.999 (.0010)	.000 (.0000)	1.000 (.0000)	.000 (.0000)	1.000 (.0000)	.000 (.0000)

Table 4: MC approximations of the rejection probabilities for the Gauss- and t_3 -copula (Panel 1) and for the Clayton- and Gumbel-copula (Panel 2) given $H_0: \rho_L \leq \rho_U$. The simulated sample size is $n = 2500$, the number of bootstrap replications corresponds to $N_B = 1000$, and the number of MC replications is $N_{MC} = 1000$. The standard errors for the approximated rejection probabilities are given in parentheses.

$H_0: \rho_L \geq \rho_U$ vs. $H_1: \rho_L < \rho_U$							
Panel 1		$\theta = 0.25$		$\theta = 0.50$		$\theta = 0.75$	
$p = q$	α	Gauss	t_3	Gauss	t_3	Gauss	t_3
0.20	0.10	.093 (.0092)	.097 (.0094)	.089 (.0090)	.090 (.0090)	.096 (.0093)	.096 (.0093)
	0.05	.044 (.0065)	.042 (.0063)	.041 (.0063)	.044 (.0065)	.044 (.0065)	.044 (.0065)
	0.01	.009 (.0030)	.011 (.0033)	.007 (.0026)	.012 (.0034)	.006 (.0024)	.013 (.0036)
0.35	0.10	.108 (.0098)	.083 (.0087)	.100 (.0095)	.111 (.0099)	.096 (.0093)	.087 (.0089)
	0.05	.053 (.0071)	.032 (.0056)	.054 (.0071)	.064 (.0077)	.048 (.0068)	.043 (.0064)
	0.01	.013 (.0036)	.007 (.0026)	.013 (.0036)	.012 (.0034)	.005 (.0022)	.013 (.0036)
0.50	0.10	.095 (.0093)	.097 (.0094)	.103 (.0096)	.094 (.0092)	.087 (.0089)	.088 (.0090)
	0.05	.054 (.0071)	.042 (.0063)	.042 (.0063)	.060 (.0075)	.040 (.0062)	.044 (.0065)
	0.01	.016 (.0040)	.007 (.0026)	.004 (.0020)	.009 (.0030)	.003 (.0017)	.015 (.0038)
Panel 2		$\rho = 0.30$		$\rho = 0.50$		$\rho = 0.70$	
$p = q$	α	Clayton	Gumbel	Clayton	Gumbel	Clayton	Gumbel
0.20	0.10	.000 (.0000)	.856 (.0111)	.000 (.0000)	.986 (.0037)	.000 (.0000)	1.000 (.0000)
	0.05	.000 (.0000)	.752 (.0137)	.000 (.0000)	.965 (.0058)	.000 (.0000)	.999 (.0010)
	0.01	.000 (.0000)	.481 (.0158)	.000 (.0000)	.870 (.0106)	.000 (.0000)	.994 (.0024)
0.35	0.10	.000 (.0000)	.964 (.0059)	.000 (.0000)	.999 (.0010)	.000 (.0000)	1.000 (.0000)
	0.05	.000 (.0000)	.926 (.0083)	.000 (.0000)	.999 (.0010)	.000 (.0000)	1.000 (.0000)
	0.01	.000 (.0000)	.790 (.0129)	.000 (.0000)	.987 (.0036)	.000 (.0000)	1.000 (.0000)
0.50	0.10	.000 (.0000)	.991 (.0030)	.000 (.0000)	1.000 (.0000)	.000 (.0000)	1.000 (.0000)
	0.05	.000 (.0000)	.974 (.0050)	.000 (.0000)	1.000 (.0000)	.000 (.0000)	1.000 (.0000)
	0.01	.000 (.0000)	.892 (.0098)	.000 (.0000)	.996 (.0020)	.000 (.0000)	1.000 (.0000)

Table 5: MC approximations of the rejection probabilities for the Gauss- and t_3 -copula (Panel 1) and for the Clayton- and Gumbel-copula (Panel 2) given $H_0: \rho_L \geq \rho_U$. The simulated sample size is $n = 2500$, the number of bootstrap replications corresponds to $N_B = 1000$, and the number of MC replications is $N_{MC} = 1000$. The standard errors for the approximated rejection probabilities are given in parentheses.

	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$
$\bar{\rho}_L$	39%	34%	34%	34%	35%
$\bar{\rho}_U$	34%	27%	25%	24%	24%
$\overline{\hat{\rho}_L - \hat{\rho}_U}$	5%	7%	9%	10%	11%
$\overline{ \hat{\rho}_L - \hat{\rho}_U }$	12%	10%	10%	11%	11%

Table 6: Average conditional Spearman's rhos, differences, and absolute differences of all 435 asset combinations for different shortfall probabilities $p = q$.

where $0 \leq \lambda \leq 1$. Further, the copula parameters θ_0, θ_1 are such that the unconditional rank-correlation coefficients of $C_{\text{Clayton}}(u, v; \theta_1)$ and $C_{\text{Gauss}}(u, v; \theta_0)$ correspond to $\rho = 0.5$. Hence, the mixed copula possesses the same unconditional rank-correlation coefficient for every λ (see the formula for ρ on p. 3).

Note that $\rho_L = \rho_U$ is true for the Gauss-copula but for the Clayton-copula it holds that $\rho_L > \rho_U$ and so the mixing parameter λ determines the degree of asymmetry given by $C_{\text{Mix1}}(u, v; \lambda, \theta_0, \theta_1)$. If one considers the two-sided hypothesis test with $H_0: \rho_L \leq \rho_U$, $\lambda = 0$ means that the null hypothesis is true whereas the alternative hypothesis holds for every $\lambda > 0$. The larger λ the more often H_0 should be rejected.

A similar result is obtained for the mixed copula

$$C_{\text{Mix2}}(u, v; \lambda, \theta_0, \theta_2) := \lambda C_{\text{Gumbel}}(u, v; \theta_2) + (1 - \lambda) C_{\text{Gauss}}(u, v; \theta_0),$$

where θ_2 is such that the rank-correlation coefficient associated with $C_{\text{Gumbel}}(u, v; \theta_2)$ once again amounts to $\rho = 0.5$. The corresponding power functions are depicted in Figure 1. Both graphs demonstrate that the hypothesis test always keeps the prescribed error probability of the first kind and the rejection probability indeed is an increasing function of the mixing parameter λ . Similar results can be obtained for other constellations of ρ and p .

5. EMPIRICAL RESULTS FOR GERMAN STOCK RETURNS

Now we consider daily returns from 1973-01-01 to 2008-11-14 of the 30 stocks of the German stock index DAX 30. The stock prices have been adjusted for dividends, splits, etc., and our analysis is based on the daily log-returns on the assets. The maximum number of observations is $n = 9359$ (trading days). Table 6 contains the sample means of the upper and lower conditional Spearman's rhos for all $\binom{30}{2} = 435$ asset combinations. Here $\bar{\rho}_L$ denotes the mean lower and $\bar{\rho}_U$ the mean upper conditional Spearman's rho, whereas $\overline{\hat{\rho}_L - \hat{\rho}_U}$ is the mean difference and $\overline{|\hat{\rho}_L - \hat{\rho}_U|}$ the mean absolute difference between $\hat{\rho}_L$ and $\hat{\rho}_U$.

The level of the shortfall probability p essentially depends on the individual needs of the investor, agent or institution. For example, a trader might be interested in the dependence structure of normal profits and losses. In that case he could choose $p = 0.5$. By contrast, a regulatory body usually focuses on extreme values and should choose a lower shortfall probability like, e.g., $p = 0.1$. For this reason we take a broad spectrum of shortfall probabilities into consideration, i.e., $p = 0.1, 0.2, \dots, 0.5$.

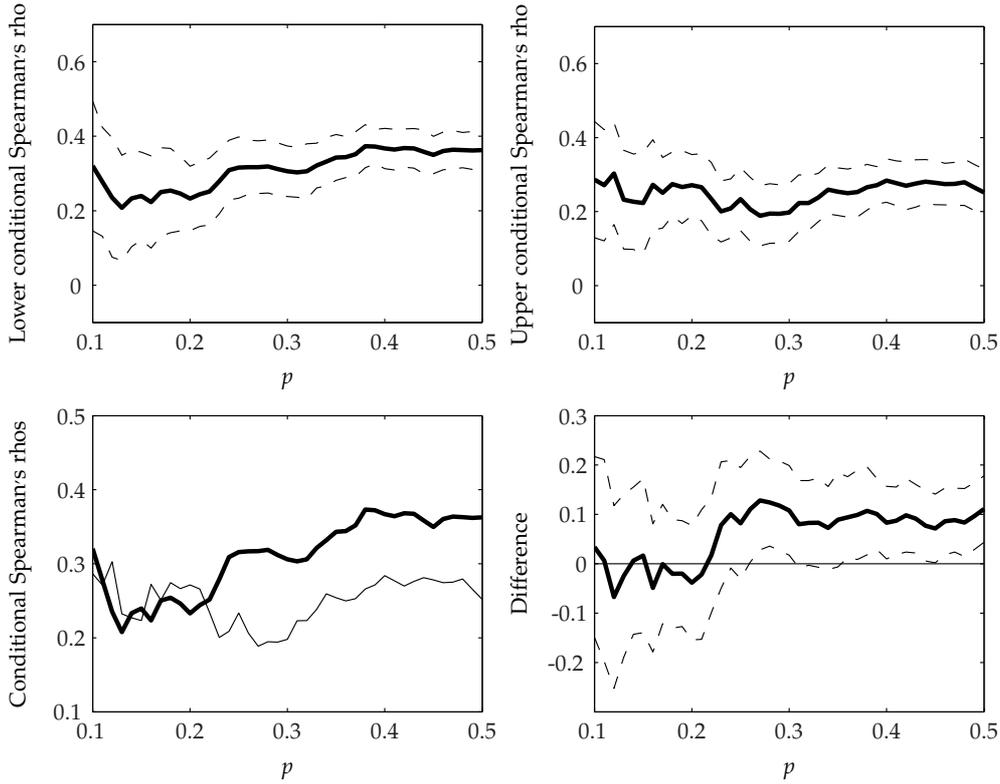


Figure 2: The upper part shows the lower (left-hand side) and upper (right-hand side) conditional Spearman's rho as a function of $p = q$ for BASF vs. Henkel ($n = 6025$). The dashed lines represent the corresponding 95%-confidence bands. In the lower left part the lower and upper conditional Spearman's rhos are shown together, where the bold line represents the lower conditional Spearman's rho. The lower right part contains the difference between the rhos and the corresponding 95%-confidence band.

It can be seen that in average the lower conditional Spearman's rhos are up to 10 points larger than the upper conditional Spearman's rhos. However, without a meaningful economic argument it is not possible to judge whether this gap is "large" or "small" and we would like to avoid such kind of statements. Instead we will discuss how much of the empirical evidence at least leads to *statistically* significant results in our hypothesis tests.

It is worth pointing out that the outcomes of the test can depend substantially on the shortfall probability p . The upper part of Figure 2 shows the lower and upper conditional Spearman's rho as a function of p for BASF vs. Henkel. The differences between the rhos (see the lower part of Figure 2) appear to be negligible if $p \leq 0.2$ but it can be very large for $p > 0.2$. The lower right part of Figure 2 indicates that it is easy to find a suitable p such that $H_0: \rho_L \leq \rho_U$ is spuriously rejected on a significance level of $\alpha = 0.05$. Therefore, to avoid a selection bias, p must be chosen *before* examining different estimates of ρ_L and ρ_U .

It is clear that the estimates $\hat{\rho}_L$ and $\hat{\rho}_U$ are different from each other for every combination of assets and we want to see whether the differences are significant in the German stock market. For this reason we apply a block-bootstrap procedure, as described in Section 3.2, to estimate the long-run variance. According to Hall et al. (1995), the optimal block size is given by $n^{\frac{1}{3}}$. In our case, the maximum number of observations is $n = 9359$ and so this rule of thumb leads to a block size of 21. Nevertheless, in view of the stylized facts of empirical finance, we think

that this choice is not conservative enough. Volatility clusters of daily log-returns are usually bigger than a trading month. Hence, we choose a block size of $b = 40$ and the number of bootstrap replications is $N_B = 1000$.

After computing the first estimate $\hat{\tau}_{LR,b}^2$, a second run with block size $b/2 = 20$ is made. This captures (approximately) the optimal block size according to Hall et al. (1995) and leads to a second estimate $\hat{\tau}_{LR,b/2}^2$ of the long-run variance. Finally, the linear combination

$$\hat{\tau}_{LR}^2 = 2\hat{\tau}_{LR,b}^2 - \hat{\tau}_{LR,b/2}^2$$

is chosen as an estimate of τ_{LR}^2 . Such a linear combination typically leads to more accurate estimates of the long-run variance (Politis, 2003).

Each rejection of the test can be interpreted as an outcome of a Bernoulli variable R_i with expected value $\pi_i \in]0, 1[$, i.e., $R_i \sim \mathcal{B}(\pi_i)$ ($i = 1, \dots, 435$). The considered hypothesis tests indeed depend on each other but, nevertheless, an unbiased estimate of $\bar{\pi} = \frac{1}{435} \sum_{i=1}^{435} \pi_i$ is given by the *rejection rate*, i.e., $\hat{\pi} = \frac{1}{435} \sum_{i=1}^{435} R_i$.

We have shown that the hypothesis tests keep the prescribed error probabilities of the first kind and are asymptotically unbiased (see Figure 1 as well as Table 4 and Table 5). This means the chosen alternative hypothesis is true for most of the 435 asset combinations if $\hat{\pi} \gg \alpha$ and in that case we say that H_1 is “true in general.” Hence, the rejection rate serves as a simple indicator for our general impression that the bivariate dependence structures of stock returns differ in bull and bear markets. Of course, a more detailed statistical analysis would require a multiple test, e.g., a Holm-Bonferroni test or another stepwise procedure (Romano and Wolf, 2005), but this is beyond our purposes.

5.1. Two-Sided Hypothesis Test

The first panel of Table 7 contains the rejection rates over all 435 asset combinations. For the shortfall probability $p = q = 0.1$ only 12% of the asset combinations exhibit significantly different Spearman’s rhos on a significance level of $\alpha = 0.1$. However, it can be seen that for all p that have been taken into consideration, the rejection rates exceed the corresponding significance levels. Especially, if p increases the rejection rates become very large and so we conclude that the lower and upper conditional rank-correlation coefficients are different from each other *in general*, i.e., for most asset combinations taken into consideration.

5.2. One-Sided Hypothesis Tests

The last rows of Panel 2 and 3 of Table 7 contain the relative numbers of asset combinations where the lower conditional Spearman’s rho exceeds the upper conditional Spearman’s rho and vice versa. For example, 66% of the asset combinations are such that $\hat{\rho}_{L,n} > \hat{\rho}_{U,n}$ under the shortfall probability $p = q = 0.1$ but only 22% of all combinations (this means $\frac{1}{3}$ of the combinations where $\hat{\rho}_{L,n}$ exceeds $\hat{\rho}_{U,n}$) are significant on the significance level $\alpha = 0.1$.

It is clear that not every combination with $\hat{\rho}_{L,n} > \hat{\rho}_{U,n}$ or $\hat{\rho}_{L,n} < \hat{\rho}_{U,n}$ can be significant. This holds in particular if the number of observations in the lower left and upper right area of the empirical copula is small. Hence, though the number of significant asset combinations might appear to be somewhat small, this neither implies that most of the null hypotheses are

Panel 1		$H_0: \rho_L = \rho_U$ vs. $H_1: \rho_L \neq \rho_U$				
α	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$	
0.10	.12	.30	.50	.63	.62	
0.05	.06	.23	.40	.55	.54	
0.01	.02	.11	.29	.38	.39	
Panel 2		$H_0: \rho_L \leq \rho_U$ vs. $H_1: \rho_L > \rho_U$				
α	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$	
0.10	.22	.41	.61	.74	.73	
0.05	.11	.29	.50	.63	.62	
0.01	.03	.14	.32	.46	.44	
$\hat{\rho}_{L,n} > \hat{\rho}_{U,n}$.66	.84	.92	.95	.98	
Panel 3		$H_0: \rho_L \geq \rho_U$ vs. $H_1: \rho_L < \rho_U$				
α	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$	
0.10	.02	.02	.00	.01	.00	
0.05	.01	.01	.00	.00	.00	
0.01	.00	.01	.00	.00	.00	
$\hat{\rho}_{L,n} < \hat{\rho}_{U,n}$.34	.16	.08	.05	.02	

Table 7: Rejection rates of the different hypothesis tests, shortfall probabilities, and significance levels. The relative numbers of asset combinations where $\hat{\rho}_L$ is larger or smaller than $\hat{\rho}_U$ are presented in the last rows of Panel 2 and Panel 3.

true nor that the differences of the lower and upper conditional rank-correlation coefficients are “small” (see the last row of Table 6).

The second panel of Table 7 reveals that the rejection rates of the hypothesis tests for various levels of p and α considerably exceed the corresponding significance levels. This effect becomes more obvious the more p increases. Hence, we can see that the hypothesis $H_1: \rho_L > \rho_U$ is true in general. By contrast, the rejection rates for the opposite tests given in Panel 3 of Table 7 are substantially smaller than the significance levels. This means there is no empirical evidence for the contrary hypothesis $H_1: \rho_L < \rho_U$.

6. CONCLUSION

Several authors have investigated the dependencies of stock returns in bull and bear markets. Pearson’s rho has been typically used as an association measure for stock returns although it depends essentially on the marginal distributions of the random variables that are taken into consideration and quantifies only the degree of linear dependence. However, one is often interested in the degree of monotone rather than linear dependence. This holds in particular if the joint distribution is highly non-linear, which is definitely the case when concentrating on the tails of stock-return distributions. So it is crucial to find a reasonable measure for the degree of monotone dependence under the condition that stock returns contemporaneously go up or down. We believe that copula theory can serve as an appropriate toolbox and suggest

Spearman's rho as a concordance measure. This is in contrast to the given literature, since most authors use conditional versions of Pearson's rho for the same purpose. Moreover, our approach is purely nonparametric. Since we do not fit specific copulas to the data or suggest specific time series models, we are able to avoid any kind of model misspecification. The finite-sample performance of the proposed hypothesis tests have been demonstrated by Monte Carlo simulation. Further, an empirical study using daily returns on stocks contained in the DAX 30 has been conducted. We think that there is sufficient evidence to support the hypothesis of different degrees of monotone dependence in bull and bear markets.

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