Tobit models for count time series

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Tobit Models for Count Time Series

Introduction

Introduction

Count process $(X_t)_{t\in\mathbb{Z}=\{...,-1,0,1,...\}}$ with range $\mathbb{N}_0 = \{0, 1, \ldots\}$ or $\{0, \ldots, N\}$ (unbounded vs. bounded counts).

Many ARMA-like model for count time series, e. g., integer-valued ARMA (**INARMA**) using "thinning operators" (McKenzie, 1985; Alzaid & Al-Osh, 1990; Du & Li, 1991; . . .), or integer-valued generalized AR conditional heteroscedasticity (**INGARCH**) via conditional mean (Ferland et al., 2006; Xu et al., 2012; Zhu, 2012; . . .), see Weiß (2018) for survey.

Common drawback: not able to explain negative ACF values!

How achieve negative ACF values?

- . . . within INGARCH framework:
  - use (highly) non-linear link function, e. g., log-linear model as in Fokianos & Tjøstheim (2011);
  - use nearly linear link function, e. g., softplus model as in Weiß et al. (2022);
- . . . within INARMA framework: ???

Our proposal (across different model classes):

Tobit approach ("Tobin's probit", see Tobin (1958)).

General Tobit approach: given process' past $X_{t-1}, \ldots$, define integer auxiliary r.v. Y_t with support $\mathbb{Z}$, define X_t by left-censoring at zero: $X_t = \max\{0,\ Y_t\}$.

Special cases:

- Tobit INGARCH model like in Weiß & Zhu (2025);
- Tobit INARMA models, work in progress together with Fukang Zhu and Hee-Young Kim.

Outline: model proposals and stochastic properties, parameter estimation and data applications.

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(Skellam) Tobit INGARCH Models

Definition & Properties

INGARCH models generally defined by linear conditional mean:

$$M_t \;=\; a_0 \;+\; \sum_{i=1}^{p} a_i\, X_{t-i} \;+\; \sum_{j=1}^{q} b_j\, M_{t-j}.$$

Then, X_t emitted by conditional count distribution,

where $M_t := E(X_t|X_{t-1},\ldots)\; > 0$ must be ensured

$\Rightarrow$ parameter constraints $a_0 > 0$ and $a_1,\ldots,a_p,b_1,\ldots,b_q \geq 0$

$\Rightarrow$ only positive ACF values.

INGARCH models generally defined by linear conditional mean:

$$M_t \;=\; \alpha_0 + \sum_{i=1}^{p} \alpha_i\, X_{t-i} + \sum_{j=1}^{q} \beta_j\, M_{t-j}.$$

Tobit INGARCH models:

Let $\mathcal{S}_{\boldsymbol{\nu}}(\mu)$ be $\mathbb{Z}$-valued random operator, where

$\mu \in \mathbb{R}$ is mean of $\mathcal{S}_{\boldsymbol{\nu}}(\mu)$ and $\boldsymbol{\nu}$ comprises further parameters.

We assume that $P\big(\mathcal{S}_{\boldsymbol{\nu}}(\mu) > 0\big) > 0$ for all μ and $\boldsymbol{\nu}$.

Then, $\quad X_t \;=\; \max\{0,\; Y_t\}\quad$ with $\quad Y_t \;=\; \mathcal{S}_{\boldsymbol{\nu}}(M_t)$.

Note: $M_t := E(Y_t | X_{t-1}, \ldots)$ allowed to become negative,

so $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q$ may be negative as well!

Skellam distribution = Poisson difference distribution, i. e.,

$Y \sim \mathsf{Sk}(\lambda_1, \lambda_2)$ iff $Y = X_1 - X_2$ with independent $X_i \sim \mathsf{Poi}(\lambda_i)$.

Its mean and variance are $\mu = \lambda_1 - \lambda_2, \quad \sigma^2 = \lambda_1 + \lambda_2 \; > |\mu|$.

Proposition: If $Y \sim \mathsf{Sk}(\lambda_1, \lambda_2)$, then partial moments

$$E\big(\max\{0, Y\}\big) \;=\; \mu\, P(Y \geq 0) \;+\; \lambda_2 \big(P(Y=0) + P(Y=1)\big)$$

and $E\big(\max\{0, Y\}^2\big) \;=$

$$(\sigma^2 + \mu^2)\, P(Y \geq 1) \;+\; \lambda_2\, \mu\, P(Y=1) \;+\; \lambda_1\, (1+\mu)\, P(Y=0).$$

Reparametrization: $\mathsf{Sk}^*(\mu, \delta)$ with $\delta := \sigma^2 - |\mu| > 0$.

Definition: Let $\mathcal{S}_\delta(\mu) \sim \mathsf{Sk}^*(\mu,\delta)$ be **Skellam operator**, then

$$X_t \;=\; \max\{0,\; Y_t\} \quad \text{with} \quad Y_t \;=\; \mathcal{S}_\delta(M_t).$$

Theorem: If $\sum_{i=1}^{p} \max\{0,\alpha_i\} + \sum_{j=1}^{q} |\beta_j| < 1$,

then STINGARCH process (X_t) exists and is stationary.

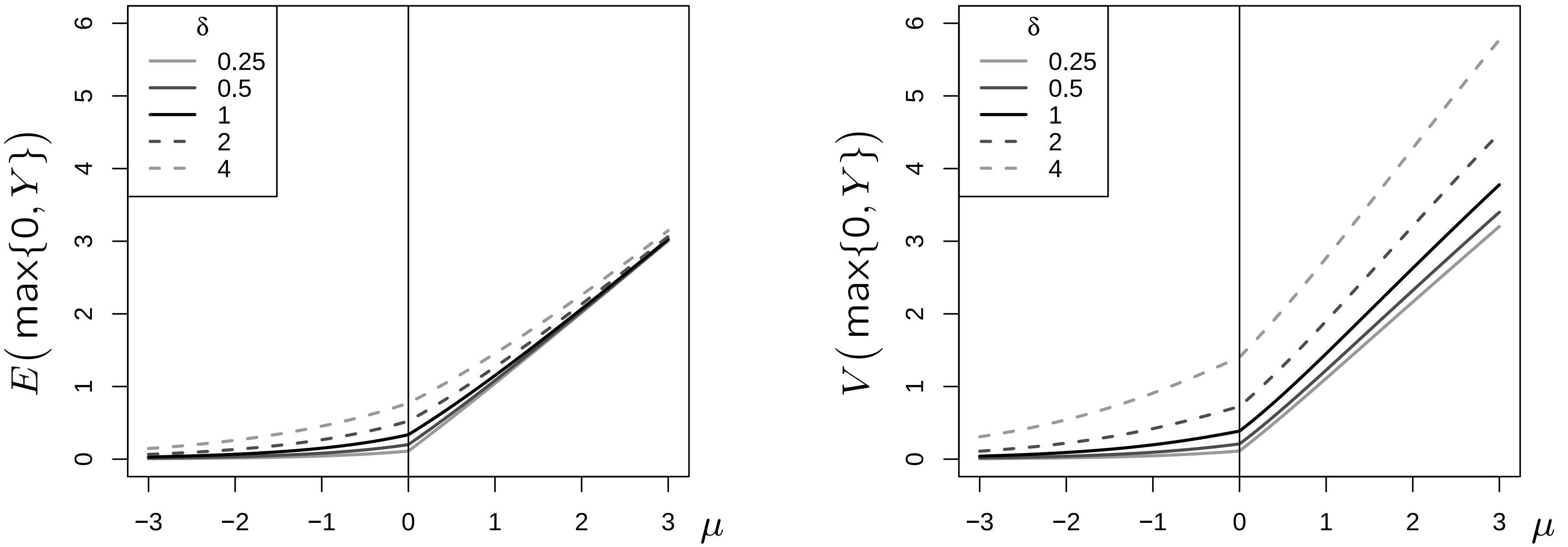
Using previous Proposition, it can be shown that

conditional mean and variance approximately piecewise linear,

i. e., STINGARCH process behaves nearly linearly

($\Rightarrow$ Yule–Walker equations for ACF)

and allows for negative ACF at same time!



In Weiß & Zhu (2025), detailed numerical study justifies following approximations if $\delta \leq 0.25$:

$$(p,q) = (1,0): \ \mu \approx \frac{\alpha_0}{1-\alpha_1}, \quad \frac{\sigma^2}{\mu} \approx \frac{I^*_{\mu,\delta}}{1-\alpha_1^2}, \quad \rho(h) \approx \alpha_1^h;$$

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$$(p,q)=(1,1):\ \mu \approx \frac{\alpha_0}{1-\alpha_1-\beta_1},$$

$$\frac{\sigma^2}{\mu} \approx I^*_{\mu,\delta}\cdot\frac{1-(\alpha_1+\beta_1)^2+\alpha_1^2}{1-(\alpha_1+\beta_1)^2},$$

$$\rho(h) \approx (\alpha_1+\beta_1)^{h-1}\,\frac{\alpha_1\big(1-\beta_1(\alpha_1+\beta_1)\big)}{1-(\alpha_1+\beta_1)^2+\alpha_1^2},$$

where $I^*_{\mu,\delta} := V\big(\max\{0,Y\}\big)\,/\,E\big(\max\{0,Y\}\big)$ for $Y\sim \mathrm{Sk}^*(\mu,\delta)$.

$\Rightarrow$ STINGARCH models as “workhorse”

for linear count processes with signed ACF values.

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(Bounded) Tobit INARMA Models

(work in progress)

Definition & Properties

We adapt the **signed INAR model** of Kim & Park (2008), which uses "signed binomial thinning" operator "$\odot$":

$$\alpha \odot X = \mathsf{sgn}(\alpha) \cdot \big(|\alpha| \circ X\big), \quad \text{where } |\alpha| \circ X \ \sim \ \mathsf{Bin}\big(X, |\alpha|\big).$$

TINAR model:

$$X_t \ = \ \max\big\{0,\ Y_t\big\}, \quad Y_t \ = \ \alpha_1 \odot X_{t-1} \ + \ldots + \ \alpha_p \odot X_{t-p} \ + \ \epsilon_t,$$

where (ϵ_t) i. i. d. count innovations.

Theorem: If all roots of polynomial equation

$$z^p - \alpha_1 z^{p-1} - \cdots - \alpha_{p-1} z - \alpha_p = 0 \quad \text{inside unit circle,}$$

then TINAR process (X_t) is stationary and ergodic.

Example: Let $\mathcal{B}_{N,\pi} \sim \text{Bin}(N,\pi)$. For TINAR(1) model,

$$P(X_t = x > 0 \mid X_{t-1}) = \sum_{i=0}^{X_{t-1}} P\big(\mathcal{B}_{X_{t-1},|\alpha_1|} = i\big) \cdot f_\epsilon\big(x - \text{sgn}(\alpha_1)\, i\big),$$

$$P(X_t = 0 \mid X_{t-1}) = \sum_{i=0}^{X_{t-1}} P\big(\mathcal{B}_{X_{t-1},|\alpha_1|} = i\big) \cdot F_\epsilon\big(- \text{sgn}(\alpha_1)\, i\big),$$

where F_ϵ is cdf of ϵ_t, and $F_\epsilon(y) = 0$ if $y < 0$.

If $\alpha_1 = -a_1 < 0$, then conditional mean nearly piecewise linear:

$$E[X_t \mid X_{t-1}] = \mu_\epsilon - a_1\, X_{t-1} + \sum_{i=0}^{X_{t-1}} P(\mathcal{B}_{X_{t-1},a_1} = i)\, E\big[(i - \epsilon_t)\, \mathbb{1}(\epsilon_t \le i)\big].$$

Numerical study shows AR(1)-like ACF also for $\alpha_1 < 0$.

TINMA model:

$$X_t \;=\; \max\{0,\; Y_t\}, \qquad Y_t \;=\; \epsilon_t + \beta_1 \odot \epsilon_{t-1} + \ldots + \beta_q \odot \epsilon_{t-q}$$
$$= \epsilon_t + \sum_{i\in\mathcal{P}} \beta_i \circ \epsilon_{t-i} - \sum_{j\in\mathcal{N}} |\beta_j| \circ \epsilon_{t-j},$$

where $\mathcal{N} = \{1 \leq i \leq q \mid \beta_i < 0\}, \quad \mathcal{P} = \{1, \ldots, q\} \setminus \mathcal{N}$.

Example: For Poi-TINMA model, Y_t has Skellam distribution:

$$Y_t \sim \mathrm{Sk}(\lambda_1, \lambda_2) \quad \text{with } \lambda_1 = \mu_\epsilon\,(1 + \textstyle\sum_{i\in\mathcal{P}} \beta_i),\; \lambda_2 = \mu_\epsilon \textstyle\sum_{j\in\mathcal{N}} |\beta_j|.$$

Together with above Proposition from Weiß & Zhu (2025), explicit expressions for mean and variance of X_t, also closed formula for lag-1 ACF of Poi-TINMA(1) model.

If (X_t) has bounded range $\{0,\ldots,N\}$,

we define **BTINARMA process** via

$$\begin{aligned} X_t &= \min\big\{N,\ \max\{0,\ Y_t\}\big\}, \\ Y_t &= \alpha_1 \odot X_{t-1} + \ldots + \alpha_p \odot X_{t-p} \\ &\quad + \epsilon_t + \beta_1 \odot \epsilon_{t-1} + \ldots + \beta_q \odot \epsilon_{t-q}, \end{aligned}$$

where innovations (ϵ_t) i. i. d. with range $\{0,\ldots,N\}$.

Theorem: If $|\alpha_1|,\ldots,|\beta_q| \in (0;1)$ and $P(\epsilon_t = y) > 0$ for all y,

then BTINARMA process (X_t) is stationary and ergodic.

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Parameter Estimation and Application

(work in progress)

Data Example

Parameter Estimation

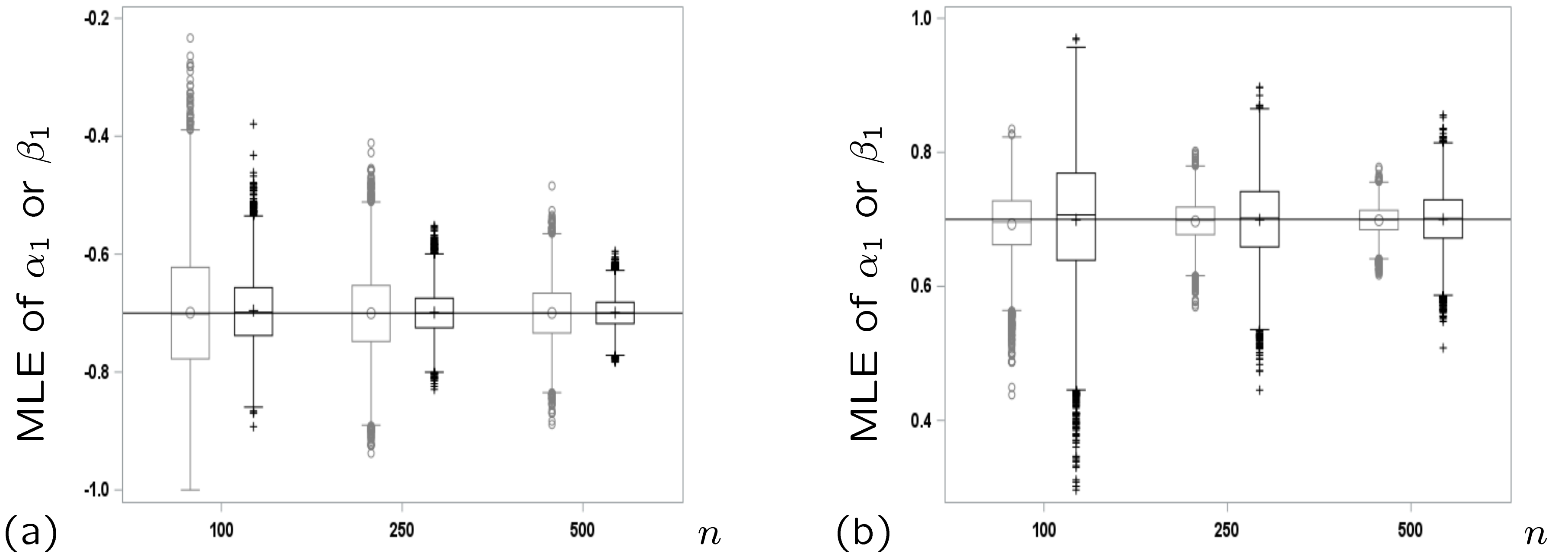
For all types of Tobit model,

maximum likelihood (ML) estimation possible:

- TINGARCH and TINAR have closed-form conditional distribution, so conditional ML estimation;
- for TINMA, forward algorithm known from Hidden-Markov models can be adapted;
- BTINARMA possesses a Markov-chain representation.

For TINGARCH and TINAR, we also analyzed censored least absolute deviations estimation (CLADE), but ML estimation superior performance.

MLE $\hat{\alpha}_1$ of TINAR(1) (left) and $\hat{\beta}_1$ of TINMA(1) (right):



(a) $(\mu_{\text{appr}}, \alpha_1) = (2.5, -0.7) = (\mu_{\text{appr}}, \beta_1)$;

(b) $(\mu_{\text{appr}}, \alpha_1) = (2.5, 0.7) = (\mu_{\text{appr}}, \beta_1)$.

Three data examples in Weiß & Zhu (2025). Here:

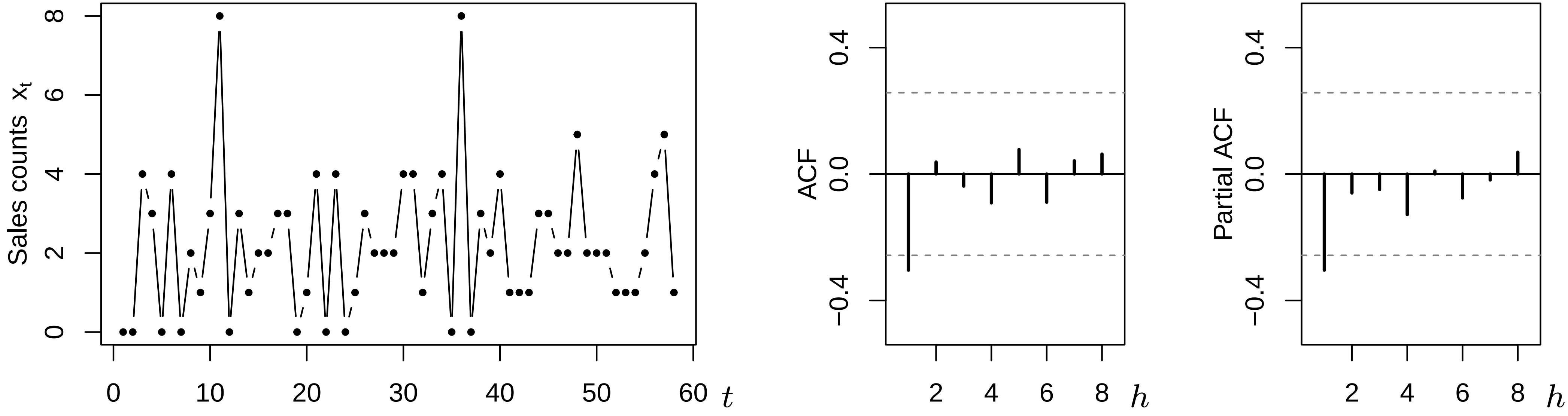
Example: Weekly Sales of Beer.

Weekly counts $x_1, \ldots, x_{58}$ of sold items of "SIERRA NEVADA STOUT (6/12 O)" in period March 28, 1996, to May 7, 1997, taken from Dominick's Data.

Sample mean ≈ 2.241 and variance ≈ 3.239 (overdispersion).

AR(1)- or MA(1)-like ACF
with negative lag-1 value ≈ -0.304: ...

Real-World Data Application



We start with three AR(1)-like candidate models:

- softplus Poi-spINARCH(1) from Weiß et al. (2022),
- Sk-TINARCH(1) from Weiß & Zhu (2025),
- and Poi-TINAR(1).

Real-World Data Application

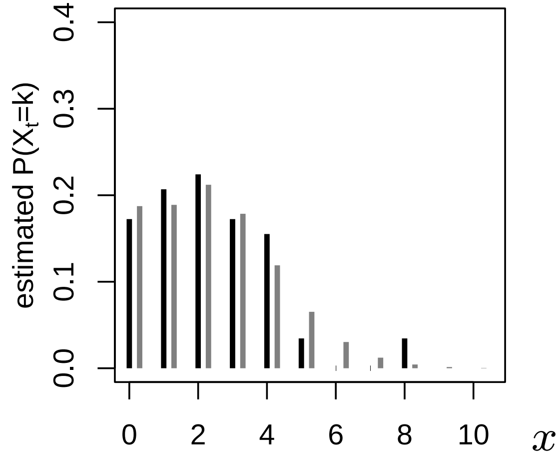
	Par. 1	Par. 2	AIC	BIC	Model fit's mean	Model fit's var	Pear. resid.'s mean	Pear. resid.'s var
Poi-spINARCH(1)	3.099 (0.398)	−0.421 (0.136)	216.3	220.4	2.181	2.651	−0.007	1.332
Sk-TINARCH(1)	3.178 (0.017)	−0.397 (0.005)	214.1	218.2	2.274	2.892	−0.005	1.224
Poi-TINAR(1)	2.915 (0.365)	−0.326 (0.147)	214.8	218.9	2.271	3.158	−0.001	0.993

spINARCH(1) outperformed by TINARCH(1) and TINAR(1) in terms of AIC and BIC.

TINAR(1) better agreement to sample mean ≈ 2.241 and variance ≈ 3.239, also superior concerning Pearson residuals.

Real-World Data Application

Poi-TINAR(1) best among AR(1)-like models.
Also marginal distribution close to sample pmf:

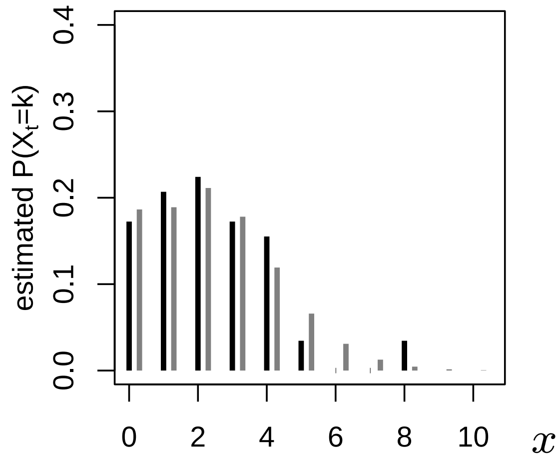


Poi-TINMA(1) similarly well concerning marginal properties,

ML estimates (s. e. in parentheses):
$\widehat{\mu}_\epsilon \approx 2.965$ (0.416), $\widehat{\beta}_1 \approx -0.261$ (0.103).

Maximized log-likelihood: -103.6.

Fitted Poi-TINMA(1) model's
mean: 2.281 (sample: 2.241),
variance: 3.188 (sample: 3.239),
lag-1 acf: -0.183 (sample: -0.304).



... but worse regarding lag-1 ACF, so TINAR(1) preferable!

Conclusions and Future Research

- Tobit approach applicable across model families.
- Tobit INGARCH and INARMA models allow for negative ACF, but still behave nearly like linear models.
- ML estimation performs well for TINGARCH and TINARMA.

Ongoing research:

- For BTINARMA models (bounded counts), still model properties need to be derived, estimation approaches not investigated so far.

Thank You for Your Interest!

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