Multiplicative Error Models for Count Time Series

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Introduction

Multiplicative error models (MEMs), see Engle (2002), Brownlees et al. (2012), Cipollini & Gallo (2023) for positively real-valued process $(Y_t)_{t\in\mathbb{Z}=\{\ldots,-1,0,1,\ldots\}}$:

$$Y_t \;=\; \mu_t \cdot \varepsilon_t,$$

where $\mu_t|\mathcal{F}_{t-1}$ deterministic, truly positive number,
ε_t positively real-valued RV with $E[\varepsilon_t \mid \mathcal{F}_{t-1}] = E[\varepsilon_t] = 1$ and $\sigma_t^2 := V[\varepsilon_t \mid \mathcal{F}_{t-1}] = V[\varepsilon_t] \in (0,\infty)$.
Observations (Y_t) satisfy

$$E[Y_t \mid \mathcal{F}_{t-1}] \;=\; \mu_t, \qquad V[Y_t \mid \mathcal{F}_{t-1}] = \sigma_t^2\,\mu_t^2.$$

Special case, see Engle & Russell (1998):

If (ε_t) i. i. d. with mean 1 and variance σ^2,

and if linear recursive scheme

$$\mu_t \;=\; a_0 \;+\; \underbrace{\sum_{i=1}^{p} a_i\, Y_{t-i}}_{\text{autoregression}} \;+\; \underbrace{\sum_{j=1}^{q} b_j\, \mu_{t-j}}_{\text{feedback}}$$

with $a_0 > 0$, $a_1, \ldots, b_q \geq 0$, and $\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j < 1$,

then **autoregressive conditional duration** (ACD) model.

Here: not positively real-valued process (Y_t),

but (positively integer-valued) count process (X_t).

Integer counterpart: **INGARCH models** by Heinen (2003), Ferland et al. (2006) for count process (X_t) with range $\mathbb{N}_0 = \{0, 1, \ldots\}$. With $M_t = E[X_t \mid \mathcal{F}_{t-1}]$, model equation

$$M_t \;=\; a_0 + \sum_{i=1}^{p} a_i\, X_{t-i} + \sum_{j=1}^{q} b_j\, M_{t-j},$$

$X_t|\mathcal{F}_{t-1}$ from conditional count distribution with mean M_t.

Default model: $X_t|\mathcal{F}_{t-1} \;\sim\; \text{Poi}(M_t)$,

but many further conditional distributions, see Weiß (2018).

Crucial difference to ACD-MEM:

no explicit multiplicative structure.

Absence of multiplicative structure because of **"multiplication problem"**, see Weiß (2018): multiplication $M \cdot \epsilon$ of count RV ϵ by real scalar $M \in (0, \infty)$ does not lead to count value anymore.

In context of **INARMA models**, so-called "thinning operators" as integer substitutes of multiplication: random operator "thinn" from $\mathbb{N}_0$ to $\mathbb{N}_0$, where $E[\text{thinn}(\epsilon) \mid \epsilon] \leq \epsilon$.

Family of **generalized thinnings** by Latour (1998):

$$\alpha \bullet_\nu \epsilon \;:=\; \sum_{j=1}^{\epsilon} Z_j \quad \text{with } E[Z_j] = \alpha \in (0,1) \text{ and } V[Z_j] = \nu > 0,$$

where Z_j i. i. d. count RV (**counting series**).

Thus, $E[\alpha \bullet_\nu \epsilon \mid \epsilon] = \alpha \cdot \epsilon \;\leq \epsilon$,

so integer substitute of multiplication in a sense.

Popular examples:

- binomial thinning $\alpha \circ \epsilon | \epsilon \;\sim \text{Bin}(\epsilon, \alpha)$;
- negative-binomial thinning $\alpha * \epsilon | \epsilon \;\sim \text{NB}\big(\epsilon, 1/(1+\alpha)\big)$;
- Poisson thinning $\alpha \bullet_\nu \epsilon | \epsilon \;\sim \text{Poi}(\epsilon \cdot \alpha)$.

Outline:

- multiplicative operators for counts;
- used to define count MEM (CMEM);
- two specific cases using compounding operator and binomial multiplicative operator, respectively;
- semi-parametric estimation methods;
- illustrative real-world data examples.

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Multiplicative Operators for Counts

Definition & Properties

"$\alpha\odot : \mathbb{N}_0 \to \mathbb{N}_0$" integer-valued **multiplicative operator** if

$$E[\alpha \odot \epsilon \mid \epsilon] \;=\; \alpha \cdot \epsilon \qquad \text{for all } \alpha > 0,$$

so multiplicative in conditional mean of $\alpha \odot \epsilon$.

It follows that $\qquad E[\alpha \odot \epsilon] = \alpha \cdot E[\epsilon]$.

Further assumptions for (conditional) variance of $\alpha \odot \epsilon$, e. g.,

$$\text{if} \quad V[\alpha \odot \epsilon \mid \epsilon] \;=\; \nu \cdot \epsilon, \quad \text{then} \quad V[\alpha \odot \epsilon] \;=\; \nu\, E[\epsilon] + \alpha^2\, V[\epsilon]$$

for some $\nu = \nu(\alpha) > 0$. For likelihood calculation or simulation, necessary to fully specify conditional distribution of $\alpha \odot \epsilon|\epsilon$.

Special case: **compounding operator**

$$\alpha \bullet_\nu \epsilon \ := \ \sum_{j=1}^{\epsilon} Z_j \qquad \text{with i. i. d. counting series } (Z_j),$$

having mean $\alpha > 0$ and variance $\nu > 0$,

where distribution of $\alpha \bullet_\nu \epsilon$ is convolution

$$P(\alpha \bullet_\nu \epsilon = k) \ = \ \sum_{l=0}^{\infty} \ P(\epsilon = l) \cdot P(\alpha \bullet_\nu l = k).$$

Important examples: Poisson or geometric counting series,

$$\alpha \bullet_\nu l \ \sim \ \begin{cases} \text{Poi}(l\,\alpha) & \text{if } Z_j \sim \text{Poi}(\alpha), \\ \text{NB}\big(l, 1/(1+\alpha)\big) & \text{if } Z_j \sim \text{NB}\big(1, 1/(1+\alpha)\big). \end{cases}$$

Bernoulli counting series $Z_j \sim \text{Bin}(1, \alpha)$ not suitable,

because then $\alpha < 1$ and $\alpha \bullet_\nu l \leq l$!

Special case: **binomial multiplicative operator**

$$\alpha \otimes \epsilon = \lfloor\alpha\rfloor \cdot \epsilon + \mathsf{Bin}\big(\epsilon,\ \alpha - \lfloor\alpha\rfloor\big) \quad \text{for any } \alpha > 0,$$

where $E[\alpha \otimes \epsilon \mid \epsilon] = \alpha \cdot \epsilon$ and

$$V[\alpha \otimes \epsilon \mid \epsilon] = (\alpha - \lfloor\alpha\rfloor)(1 - \alpha + \lfloor\alpha\rfloor)\,\epsilon =: \nu(\alpha) \cdot \epsilon.$$

Distribution determined by

$$\mathsf{pgf}_{\alpha\otimes\epsilon}(u) = \mathsf{pgf}_{\epsilon}\Big(u^{\lfloor\alpha\rfloor}\big(1 - (\alpha - \lfloor\alpha\rfloor) + (\alpha - \lfloor\alpha\rfloor)\,u\big)\Big).$$

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MEMs for Count Time Series

Definition & Properties

(X_t) with range $\mathbb{N}_0$ is **CMEM** if $\quad X_t \;=\; M_t \odot \epsilon_t,$

where $M_t|\mathcal{F}_{t-1}$ deterministic truly positive number.

(ϵ_t) i. i. d. count RV with $E[\epsilon_t] = 1$ and $V[\epsilon_t] = \sigma^2$.

Operator "$\odot$" executed independently of $\mathcal{F}_{t-1}$ and ϵ_t.

We have $E[X_t \mid \mathcal{F}_{t-1}] \;=\; M_t$. **INGARCH**$(p,q)$-**CMEM** if

$$M_t \;=\; f(X_{t-1},\ldots,X_{t-p},\; M_{t-1},\ldots,M_{t-q}), \quad p \geq 1,\; q \geq 0.$$

If "$\odot$" also satisfies $V[\alpha \odot \epsilon \,|\, \epsilon] = \nu(\alpha) \cdot \epsilon$, then

$$V[X_t \mid \mathcal{F}_{t-1}] \;=\; \nu(M_t) + \sigma^2\, M_t^2.$$

Establish existence using Doukhan & Neumann (2019).

Special case: **compounding CMEM** $X_t \;=\; M_t \bullet_\nu \epsilon_t$,

satisfying

$$E[X_t \mid \mathcal{F}_{t-1}] \;=\; M_t, \qquad V[X_t \mid \mathcal{F}_{t-1}] \;=\; \nu(M_t) + \sigma^2 M_t^2.$$

Poi-counting series: $\nu(M_t) = M_t$; NB-c.s.: $\nu(M_t) = M_t(1 + M_t)$.

In Weiß & Zhu (2023), we prove:

- first-order stationary iff $\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j < 1$;
- conditions for second-order stationarity;
- moment formulae, YW-type equations for ACF;
- CP-INGARCH process stationary and ergodic.

Possible competitor by Aknouche & Scotto (2023):

multiplicative thinning-based INGARCH model ($m \in \mathbb{N}$ specified)

$$X_t = \lambda_t \cdot \epsilon_t, \qquad \lambda_t = 1 + \omega \circ m + \sum_{i=1}^{q} \alpha_i \circ Y_{t-i} + \sum_{j=1}^{p} \beta_j \circ \lambda_{t-j}.$$

Relation of our INGARCH-CMEM to ordinary INGARCH:

- $X_t \;=\; M_t \bullet_\nu \epsilon_t$ with Poi-counting series:
  if $\epsilon_t \equiv 1$, then Poi-INGARCH of Ferland et al. (2006).
- We propose **semi-parametric INGARCH model:**
  ϵ_t with finite support, parameters $p_{\epsilon,2}, p_{\epsilon,3}, \ldots$ chosen
  subject to constraint $\sum_{l\geq 2} l \cdot p_{\epsilon,l} < 1$,
  where $p_{\epsilon,0} = \sum_{l\geq 2}(l-1) \cdot p_{\epsilon,l}$ and $p_{\epsilon,1} = 1 - p_{\epsilon,0} - \sum_{l\geq 2} p_{\epsilon,l}$.

Special case: **binomial-multiplic. CMEM** $X_t \;=\; M_t \otimes \epsilon_t$,

satisfying $E[X_t \mid \mathcal{F}_{t-1}] \;=\; M_t$ and

$$V[X_t \mid \mathcal{F}_{t-1}] \;=\; (M_t - \lfloor M_t \rfloor)(1 - \;M_t + \lfloor M_t \rfloor) + \sigma^2 M_t^2.$$

In Weiß & Zhu (2023), we prove:

- inequality for variance:

$$\mu^2\sigma^2 + (\sigma^2+1)\,V[M_t] \;\leq\; V[X_t] \;\leq\; 0.25 + \mu^2\sigma^2 + (\sigma^2+1)\,V[M_t],$$

  where $V[M_t] \;=\; \sum_{i=1}^{p} a_i\,\gamma_X(i) \;+\; \sum_{j=1}^{q} b_j\,\gamma_M(j)$;

- if $\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j < 1$,

  then weakly dependent stationary and ergodic solution.

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Parameter Estimation and Model Diagnostics

Approaches & Simulations

Parameter Estimation and Model Diagnostics

In Weiß & Zhu (2023), we derive

several **semi-parametric estimation** approaches:

- quasi-maximum likelihood estimation (QMLE), namely Poi-QMLE, NB-QMLE, and exponential QMLE,
- two-stage weighted least squares estimation (2SWLSE).

QMLE relies only on conditional mean M_t,

estimation of σ^2 by utilizing conditional variance.

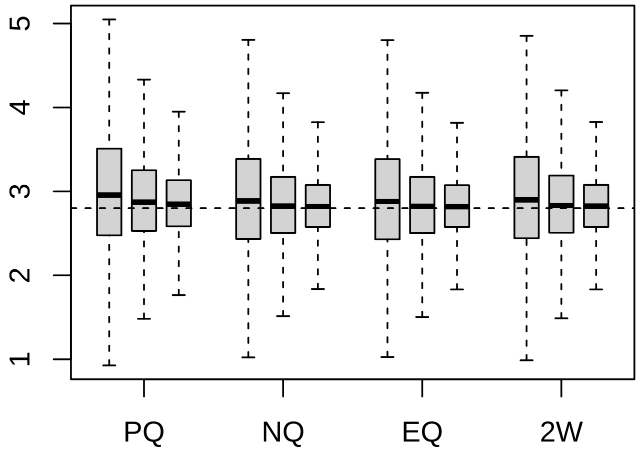
2SWLSE uses both conditional mean and variance.

We use residual-based **model diagnostics**.

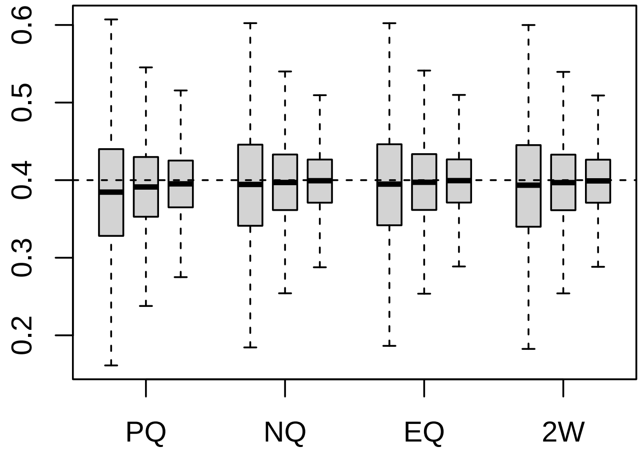
Simulations for finite-sample performance (also misspecific.).

Parameter Estimation and Model Diagnostics

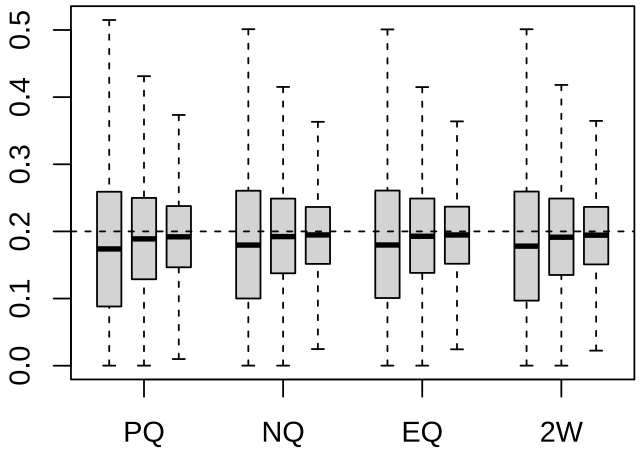
Estimation of $a_0 = 2.8$



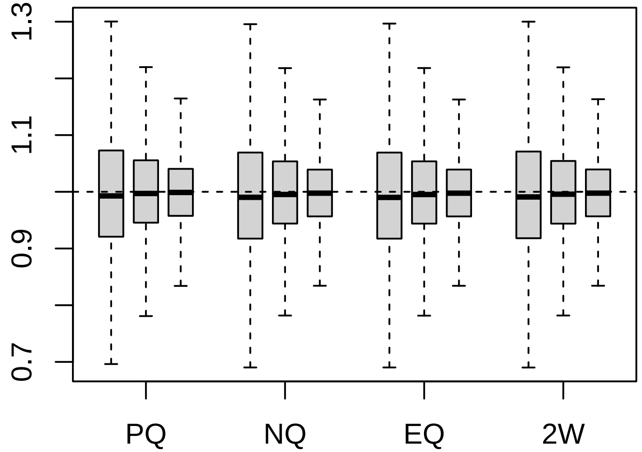
Estimation of $a_1 = 0.4$



Estimation of $b_1 = 0.2$



Estimation of $\sigma^2 = 1$



Real-World Data Examples

Illustration

Real-World Data Examples

For illustration, we discuss **two data examples** from literature:

- Escherichia coli (Ecoli) disease counts from Aknouche & Scotto (2023),
- Wausau Paper Corporation (WPP) transaction counts from Aknouche et al. (2022).

Fit INGARCH$(1,1)$-CMEM with Poi- and Bin-counting series.

Both in-sample performance and predictive performance superior to competing models.

PQ-fitted CMEMs show best predictive performance.

Detailed discussion in Weiß & Zhu (2023).

Conclusions and Future Research

- Novel framework for transferring well-established MEMs for real-valued time series to case of count time series.
- Proposed CMEMs use multiplicative operator for counts.
- Existence of stationary and ergodic solutions, moment properties, semi-parametric model estimation.

Future research:

- CMEMs using other multiplicative operator (e. g., ZIP), or non-linear model specifications (logit, softplus, . . .).
- Semi-parametric ML estimation for INGARCH models, like semi-parametric INAR approach of Drost et al. (2009).

Thank You for Your Interest!

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