Stein EWMA Control Charts for Count Processes

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Introduction

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Monitoring of (possibly autocorrelated) **count process** $(X_t)_{t\in\mathbb{N}}$, i. e., X_t have quantitative range contained in $\mathbb{N}_0 = \{0, 1, \ldots\}$ (e. g., counts of defects, counts of hospital admissions, etc.).

We distinguish between

- **unbounded counts** having full $\mathbb{N}_0$ as range (e. g., Poisson (Poi) or negative binomial (NB) counts);
- **bounded counts** having range $\{0, \ldots, n\}$ with $n \in \mathbb{N}$ (e. g., binomial (Bin) counts).

Many control charts for monitoring count processes proposed, see Weiß (2015), Alevizakos & Koukouvinos (2020); here: . . .

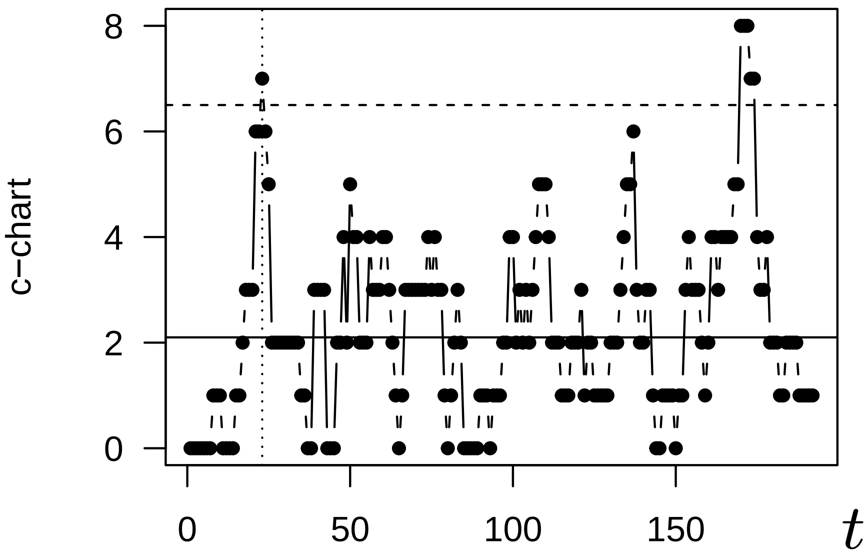
Exponentially weighted moving-average (EWMA) chart for counts, considered by many authors, among others by Borror et al. (1998), Morais & Knoth (2020), Anastasopoulou & Rakitzis (2022). **Default EWMA chart** plots

$$Z_0 = \mu_0, \quad Z_t \;=\; \lambda \cdot X_t \;+\; (1-\lambda) \cdot Z_{t-1} \quad \text{for } t = 1, 2, \ldots$$

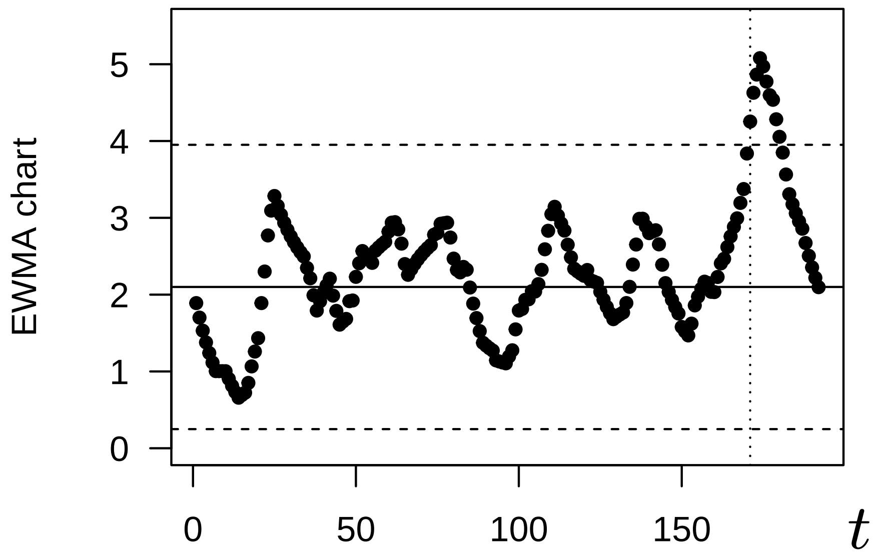
against specified lower and upper control limit (LCL and UCL), where $\mu_0 > 0$ in-control mean of (X_t).
Smoothing parameter $\lambda \in (0; 1]$ controls strength of memory (the smaller λ, the stronger the memory).
Reduces to memory-less **c- or np-chart** if $\lambda = 1$.

Illustrative example: (to be discussed later)

Upper-sided c-chart:



Two-sided EWMA chart:



EWMA chart with $\lambda = 0.1$.

CLs as dashed lines, solid center line.

First alarm at dotted line ($t = 23$ vs. $t = 171$).

Existing count EWMA charts solely designed
to detect shifts in process mean. Here,
focus on “more sophisticated” **out-of-control scenarios:**

- mean changes together with further distributional changes,
- or purely distributional changes (not affecting the mean).

These distributional changes might be
increases or decreases in dispersion compared to in-control model (overdispersion or underdispersion, respectively),
or excessive number of zero counts (zero inflation).

Basic idea: Many common (count) distributions characterized by **Stein identity** (Stein, 1972), i. e., moment identity that has to hold for large class of functions, see Sudheesh & Tibiletti (2012) for details and references.

Recently: Stein GoF-tests for i. i. d. counts (Weiß et al., 2023) and for count time series (Aleksandrov et al., 2022a,b).

$\Rightarrow$ **Utilize Stein identities for control charts**.

In Weiß (2022), idea tried out for i. i. d. Poisson counts, achieved chart performance was quite appealing.

Here: Stein control charts on much broader scale, namely various different count distributions and also time series data.

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Count Models and Stein Identities

Brief Overview

We consider either i. i. d. Poi-, NB-, or Bin-counts, or AR(1)-type counts with Poi-, NB-, or Bin-*marginal* distribution, where thinning operator as integer substitute of multiplication:

(see Weiß, 2008)

- **Poi-INAR**(1) **process** (integer AR)

$$X_t \;=\; \rho\circ X_{t-1}+\epsilon_t \quad \text{with i. i. d. } \epsilon_t\sim \text{Poi}\big(\mu(1-\rho)\big);$$

- **NB-IINAR**(1) **process** (iterated-thinning INAR)

$$X_t \;=\; \rho\circledast_\pi X_{t-1}+\epsilon_t \quad \text{with i. i. d. } \epsilon_t\sim \text{NB}(\nu,\pi),\ \pi=\tfrac{\nu}{\mu(1-\rho)+\nu};$$

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- **BinAR**(1) **process**

$$X_t = \alpha \circ X_{t-1} + \beta \circ (n - X_{t-1}) \quad \text{with } \beta = (1-\rho)\,\tfrac{\mu}{n},\ \alpha = \beta + \rho.$$

Here, binomial thinning "$\circ$" defined by $\theta \circ X | X \sim \text{Bin}(X, \theta)$, iterated thinning as $\rho \circledast_\pi X = \sum_{i=1}^{(\pi\rho)\circ X} Y_i$, where $Y_i \sim 1 + \text{NB}(1, \pi)$.

Crucial point: all models are stationary Markov chains with ACF $\rho(h) = \rho^h$ and marginal distributions Poi(μ), NB($\nu, \frac{\nu}{\nu+\mu}$), or Bin($n, \mu/n$). Parameter ρ controls extent of serial dependence, where $\rho \to 0$ leads to i. i. d. counts.

Relevant **Stein identities**, see Sudheesh & Tibiletti (2012):

- $X \sim \text{Poi}(\mu)$ iff $\quad E\big[X\,f(X)\big] \;=\; \mu\,E\big[f(X+1)\big]$,
- $X \sim \text{NB}(\nu, \frac{\nu}{\nu+\mu})$ iff

$$(\nu+\mu)\,E\big[X\,f(X)\big] \;=\; \mu\,E\big[(\nu+X)\,f(X+1)\big],$$

- $X \sim \text{Bin}(n, \mu/n)$ iff

$$(n-\mu)\,E\big[X\,f(X)\big] = \mu\,E\big[(n-X)\,f(X+1)\big],$$

hold for all bounded functions $f: \mathbb{N}_0 \to \mathbb{R}$

(f not constant on $\mathbb{N}_0$, not identical zero on $\mathbb{N}$).

f can be chosen arbitrarily $\quad\Rightarrow\quad$ **weight function**.

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Stein EWMA Charts for Counts

Proposed Approach

The Stein identities depend on three types of moment: mean μ, moment $E\big[X\,f(X)\big]$, and a moment involving $f(X+1)$.

Idea: Derive statistic by solving identities in certain way, substitute population moments by appropriate sample moments.

Initial study by Weiß (2022): two types of EWMA chart, where so-called "ABC-EWMA chart" clearly superior.

In what follows, we solely focus on this "ABC construction".

Statistics Z_t^{S} plotted against $\mathsf{LCL} < 1 < \mathsf{UCL}$, where Z_t^{S} expected close to 1 under in-control assumptions.

For all three **Stein EWMA charts**, we compute

$$A_0 = E_0\big[X\, f(X)\big], \qquad A_t \;=\; \lambda\cdot X_t\, f(X_t) \;+\; (1-\lambda)\cdot A_{t-1},$$
$$C_0 = \mu_0, \qquad C_t \;=\; \lambda\cdot X_t \;+\; (1-\lambda)\cdot C_{t-1}, \qquad \text{for } t=1,2,\ldots$$

$E_0[\cdot]$ expresses expectation with respect to in-control model.

Furthermore, we compute for

- **Poi**(μ_0) **in-control model:**

$$B_0 = E_0\big[f(X+1)\big], \quad B_t \;=\; \lambda\cdot f(X_t+1) \;+\; (1-\lambda)\cdot B_{t-1},$$
$$Z_0^{\mathrm{S}} = 1, \qquad Z_t^{\mathrm{S}} \;=\; \frac{A_t}{B_t\, C_t}, \qquad \text{for } t=1,2,\ldots;$$

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- **NB**$(\nu, \frac{\nu}{\nu+\mu_0})$ **in-control model:**

$$B_0 = E_0\big[(\nu + X)\, f(X+1)\big],$$
$$B_t \;=\; \lambda\cdot(\nu + X_t)\, f(X_t+1) \;+\; (1-\lambda)\cdot B_{t-1},$$
$$Z_0^{\mathsf{S}} = 1, \quad Z_t^{\mathsf{S}} \;=\; \frac{(\nu + C_t)\, A_t}{B_t\, C_t}, \qquad \text{for } t = 1, 2, \ldots;$$

- **Bin**$(n, \mu_0/n)$ **in-control model:**

$$B_0 = E_0\big[(n - X)\, f(X+1)\big],$$
$$B_t \;=\; \lambda\cdot(n - X_t)\, f(X_t+1) \;+\; (1-\lambda)\cdot B_{t-1},$$
$$Z_0^{\mathsf{S}} = 1, \quad Z_t^{\mathsf{S}} \;=\; \frac{(n - C_t)\, A_t}{B_t\, C_t}, \qquad \text{for } t = 1, 2, \ldots$$

f chosen in view of the anticipated out-of-control scenario,

most weight on those regions of $\mathbb{N}_0$

where strongest departures from in-control model, e. g.,

- **linear weights** $f(x) = |x-1|$ for uncovering overdispersion,
- **root weights** $f(x) = |x-1|^{1/4}$ for zero inflation.

As we shall see, detection of underdispersion quite demanding,

possible first idea:

- **inverse weights** $f(x) = 1/(x+1)$ for underdispersion.

If not single relevant out-of-control scenario,

then run multiple Stein EWMA charts in parallel.

Design and performance evaluation of Stein EWMA charts based on average run length (ARL) simulations ($R = 10^4$).

Although many different ARL concepts (see Knoth, 2006), initial study by Weiß (2022) showed that Stein EWMA charts roughly same performance for early and late process changes. Thus, subsequent analyses restrict to zero-state ARLs.

Since not one specific out-of-control scenario as in Morais & Knoth (2020), but broad variety:
We use symmetric CLs $\mu_0 \mp L$ for ordinary EWMA and $1 \mp L$ for Stein EWMA charts, and unique choice $\lambda = 0.10$.

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Simulation-based Performance Analyses

Selected Results

In-control: Bin-counts with $n = 10$, $\mu_0 \in \{2, 5\}$.

Out-of-control: mean shifts ∓ 0.25, BB,ZIB with $I_B = 5/3$.

μ_0	$\mu =$	$\mu_0 - 0.25$	μ_0	$\mu_0 + 0.25$	$\mu_0 - 0.25$	μ_0	$\mu_0 + 0.25$	$\mu_0 - 0.25$	μ_0	$\mu_0 + 0.25$
(a)	i. i. d. counts									CL L in italic font.
		EWMA		*0.7805*	EWMA^S	$\lvert x-1 \rvert$	*0.534*	EWMA^S	$\lvert x-1 \rvert^{1/4}$	*0.4235*
2	ZIB	69.2	87.9	51.6	22.7	26.1	29.2	16.6	19.1	21.9
	Bin	171.5	**370.2**	99.3	240.2	**369.5**	550.6	191.5	**370.6**	671.0
	BB	71.5	90.0	51.8	25.9	29.2	33.3	30.8	40.5	55.6
		EWMA		*0.974*	EWMA^S	$\lvert x-1 \rvert$	*0.2115*	EWMA^S	$\lvert x-1 \rvert^{1/4}$	*0.0511*
5	ZIB	69.2	87.9	51.6	18.9	19.9	20.6	12.9	14.0	15.5
	Bin	162.5	**369.5**	164.1	307.2	**370.1**	443.4	311.5	**369.5**	417.2
	BB	66.7	88.2	66.3	25.1	26.9	29.0	27.4	28.9	30.3

sole mean change | pure distrib. change | zero inflation / overdispersion

In-control: Bin-counts with $n = 10$, $\mu_0 \in \{2, 5\}$.

Out-of-control: mean shifts ∓ 0.25, BB,ZIB with $I_{\mathrm{B}} = 5/3$.

μ_0	$\mu =$	$\mu_0 - 0.25$	μ_0	$\mu_0 + 0.25$	$\mu_0 - 0.25$	μ_0	$\mu_0 + 0.25$	$\mu_0 - 0.25$	μ_0	$\mu_0 + 0.25$
(b)	AR(1) counts with $\rho = 0.5$									CL L in italic font.
		EWMA		*1.191*	EWMA^{S}	$\lvert x-1 \rvert$	*0.639*	EWMA^{S}	$\lvert x-1 \rvert^{1/4}$	*0.568*
2	ZIB	77.9	73.8	55.8	22.9	24.3	25.3	23.8	26.9	30.0
	Bin	384.8	**370.1**	158.0	247.1	**369.7**	554.4	211.9	**371.2**	634.3
	BB	105.0	105.7	77.2	33.0	39.5	47.0	44.7	58.9	77.1
		EWMA		*1.493*	EWMA^{S}	$\lvert x-1 \rvert$	*0.225*	EWMA^{S}	$\lvert x-1 \rvert^{1/4}$	*0.0528*
5	ZIB	30.8	29.9	28.0	7.9	7.7	7.3	7.1	6.9	6.5
	Bin	258.0	**369.1**	257.3	316.7	**370.9**	424.1	293.9	**370.9**	421.8
	BB	86.3	96.1	85.9	35.3	37.6	39.5	37.9	39.4	40.7

EWMA columns	EWMA^S $\lvert x-1 \rvert$ columns	EWMA^S $\lvert x-1 \rvert^{1/4}$ columns
sole mean change	pure distrib. change	zero inflation overdispersion

In-control: NB-counts with $\mu_0 \in \{2, 5\}$, $I_P = 5/3$.

Out-of-control: mean shifts ∓ 0.25, ZIP with $I_P = 5/3$, overdispersed NB ("oNB") with $I_P = 5/2$.

μ_0	$\mu =$	$\mu_0 - 0.25$	μ_0	$\mu_0 + 0.25$	$\mu_0 - 0.25$	μ_0	$\mu_0 + 0.25$	$\mu_0 - 0.25$	μ_0	$\mu_0 + 0.25$
(a)	i. i. d. counts									CL L in italic font.
		EWMA		*1.156*	EWMA^S	$\lvert x-1 \rvert$	*0.349*	EWMA^S	$\lvert x-1 \rvert^{1/4}$	*0.3146*
2	ZIP	506.6	462.0	149.7	139.4	257.2	481.5	54.2	81.0	124.9
	NB	605.8	**370.7**	133.1	172.1	**370.9**	892.2	154.0	**369.9**	1001.2
	oNB	171.9	135.1	75.6	45.2	67.2	103.8	52.2	86.9	163.2
		EWMA		*1.805*	EWMA^S	$\lvert x-1 \rvert$	*0.1554*	EWMA^S	$\lvert x-1 \rvert^{1/4}$	*0.0883*
5	ZIP	342.6	407.5	261.9	116.2	143.9	170.2	22.7	24.7	26.8
	NB	444.7	**370.8**	205.3	267.3	**369.8**	522.7	260.0	**370.1**	537.2
	oNB	130.2	124.7	94.8	42.9	51.3	59.1	55.5	68.7	87.3

sole mean change	pure distrib. change	zero inflation / overdispersion

In-control: Poi-counts with $\mu_0 \in \{2, 5\}$.

Out-of-control: mean shifts ∓ 0.25,

Good with $I_P = 3/4$ (first row) and $I_P = 1/2$ (second row).

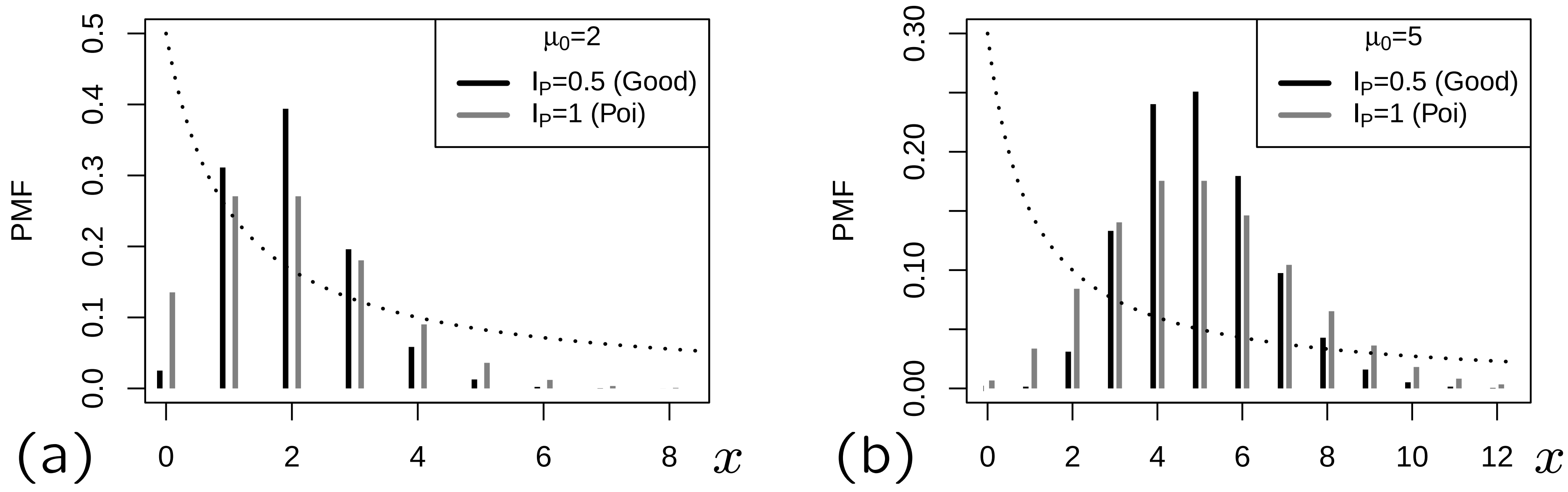
μ_0	$\mu =$	$\mu_0 - 0.25$	μ_0	$\mu_0 + 0.25$	$\mu_0 - 0.25$	μ_0	$\mu_0 + 0.25$	$\mu_0 - 0.25$	μ_0	$\mu_0 + 0.25$
(a)	i. i. d. counts									CL L in italic font.
		EWMA		*0.877*	EWMA^{S}	$1/(x+1)$	*0.223*	EWMA^{S}	$p_{\text{P}}(x+2)$	*0.608*
2	Poi	252.6	**369.1**	106.1	274.6	**368.9**	470.8	538.9	**370.3**	271.7
	Good	622.4	948.9	158.5	96.9	142.3	249.5	90.7	71.4	60.5
	Good	3611.8	6346.9	380.6	32.3	42.2	63.0	29.1	26.2	24.3
		EWMA		*1.388*	EWMA^{S}	$1/(x+1)$	*0.1775*	EWMA^{S}	$p_{\text{P}}(x+2)$	*0.293*
5	Poi	309.9	**371.4**	185.1	352.9	**370.5**	398.1	526.1	**368.7**	268.9
	Good	1006.0	1015.9	327.2	$> 10^4$	$> 10^4$	$> 10^4$	228.4	149.3	106.4
	Good	8154.5	7882.4	1168.9	$> 10^4$	$> 10^4$	$> 10^4$	52.5	40.1	33.2
		sole mean change				pure distrib. change			increasing underdispersion	

Plots of probability mass function (PMF)

for Good with $I_\mathrm{P} = 0.5$ vs. Poi,

for (a) $\mu_0 = 2$ and (b) $\mu_0 = 5$.

Dotted line proportional to $1/(x+1)$.



An Illustrative Data Application

Emergency Counts

Weiß & Testik (2015) analyze large set of daily counts (per 5-min interval from 08:00:00 to 23:59:59) on registrations in emergency department of children's hospital. Full set of time series covers period from February 13 to August 13, 2009.

Phase-I analysis: sixteen time series from February 13 to 28
$\Rightarrow$ **in-control model:** Poi-INAR(1) with $\mu_0 = 2.1$, $\rho_0 = 0.78$.

Weiß & Testik (2015) recognized that most Phase-II series do not contradict in-control model, but few unusual days.

Now Phase-II analysis using novel Stein EWMA charts.

Chart designs: **c-chart** only one-sided, LCL $= 0$ and UCL $=$ 6 with ARL$_0 \approx 326.2$ (so $X_t > 6$ causes alarm).

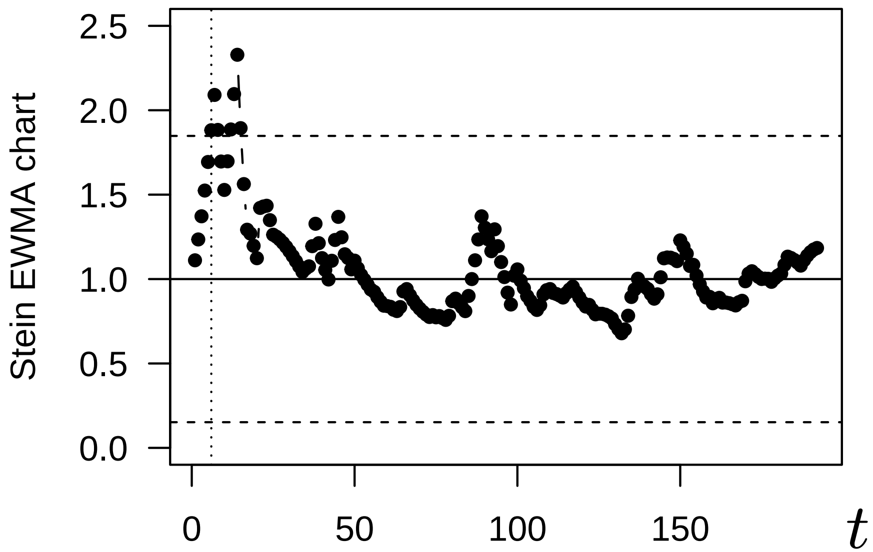
EWMA charts ($\lambda = 0.1$) truly two-sided:

- EWMA with $L = 1.851$ and ARL$_0 \approx 370.3$;
- Stein EWMA with $f(x) = |x-1|$, $L = 0.848$, ARL$_0 \approx 370.5$,
- with $f(x) = |x-1|^{1/4}$, $L = 0.829$, ARL$_0 \approx 370.5$,
- with $f(x) = 1/(x+1)$, $L = 0.2994$, ARL$_0 \approx 370.5$,
- with $f(x) = p_{\mathsf{P}}(x+2)$, $L = 0.9594$, ARL$_0 \approx 370.2$.

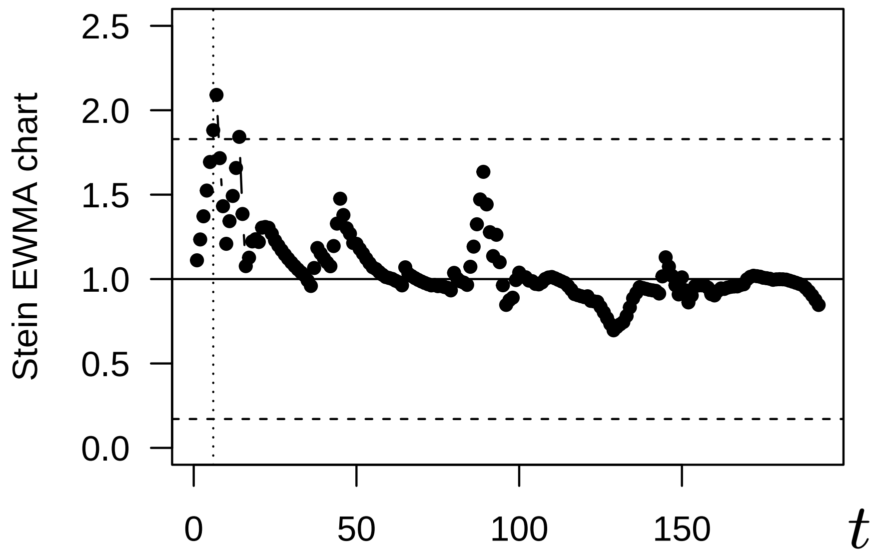
Emergency counts from March 28, 2009:

c-chart and ordinary EWMA on Slide 5. No alarm for Stein EWMA with $f(x) = 1/(x+1)$ and $f(x) = p_P(x+2)$.

S-EWMA, $f(x) = |x-1|$:



S-EWMA, $f(x) = |x-1|^{1/4}$:

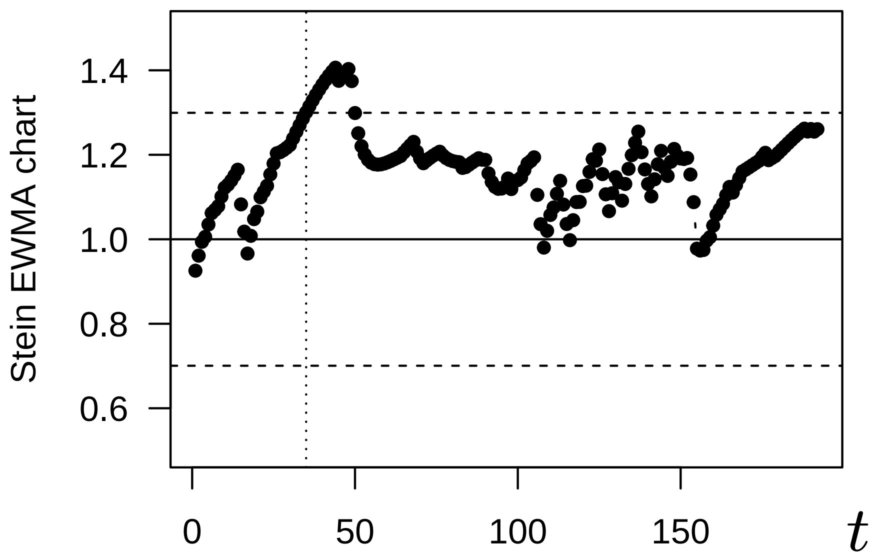


$\Rightarrow$ very early alarm at $t = 6$, so overdispersion and zero inflation.

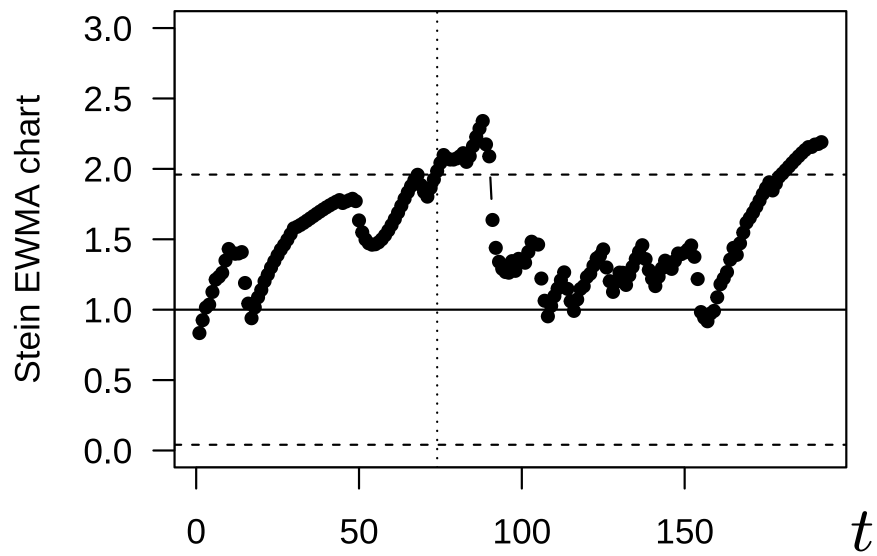
Emergency counts from July 16, 2009:

No alarm for c-chart, ordinary EWMA, also no alarm for Stein EWMA with $f(x) = |x-1|$ and $f(x) = |x-1|^{1/4}$.

S-EWMA, $f(x) = 1/(x+1)$:



S-EWMA, $f(x) = p_{\mathsf{P}}(x+2)$:



$\Rightarrow$ early alarms at $t = 35$ and 74, resp., so underdispersion.

Conclusions

- EWMA-type control charts derived from Stein identities.
- Stein EWMA charts flexibly adapted to various out-of-control scenarios, where underdispersion most demanding.
- Appealing ARL performance if adequate weight function.
- If running multiple Stein EWMA charts in parallel, targeted diagnosis possible (like in data example).
- **Future research:**
  - Stein *CUSUM* charts for count processes?
  - Stein charts for continuously distributed data?
  - . . . ?

Thank You for Your Interest!

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