Approximately Linear INGARCH Models for Spatio-Temporal Counts

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Approximately Linear INGARCH Models for Count Time Series

Introduction

INGARCH Models for Count Time Series

Count processes (X_t): range contained in $\mathbb{N}_0$,

i. e., set of *non-negative* integers (see Weiß, 2018).

Distinguish **two cases:**

- **unbounded** counts (X_t), i. e., full $\mathbb{N}_0$ as range;
- **bounded** counts (X_t), i. e., upper bound $n \in \mathbb{N}$,

  so range $\{0,\ldots,n\}$ also bounded from above!

(Exactly linear) **INGARCH model** (Ferland et al., 2006)

for **unbounded** counts (X_t): Let $M_t = E[X_t|\mathcal{F}_{t-1}]$, then

$$M_t \;=\; a_0 \;+\; \sum_{i=1}^{p} a_i\, X_{t-i} \;+\; \sum_{j=1}^{q} b_j\, M_{t-j}.$$

Counts $X_t|\mathcal{F}_{t-1}$ emitted by, for example, Poi(M_t).

(Exactly linear) **BINGARCH model** (Ristić et al., 2016)

for **bounded** counts (X_t): Let $P_t = \frac{1}{n} M_t$, then

$$P_t \;=\; a_0 \;+\; \sum_{i=1}^{p} a_i\, X_{t-i}/n \;+\; \sum_{j=1}^{q} b_j\, P_{t-j}.$$

Counts $X_t|\mathcal{F}_{t-1}$ emitted by, for example, $\text{Bin}(n, P_t)$.

Problem: *In both cases, rather limiting parameter constraints that prevent negative parameter and ACF values.*

Possible solution: approximately linear (B)INGARCH models by Weiß et al. (2022), Weiß & Jahn (2021), with negative ACF values while (nearly) linear model structure.

Solution by Weiß et al. (2022) for unbounded counts:

softplus INGARCH model with response function $f = \mathrm{sp}_c$.

Softplus function

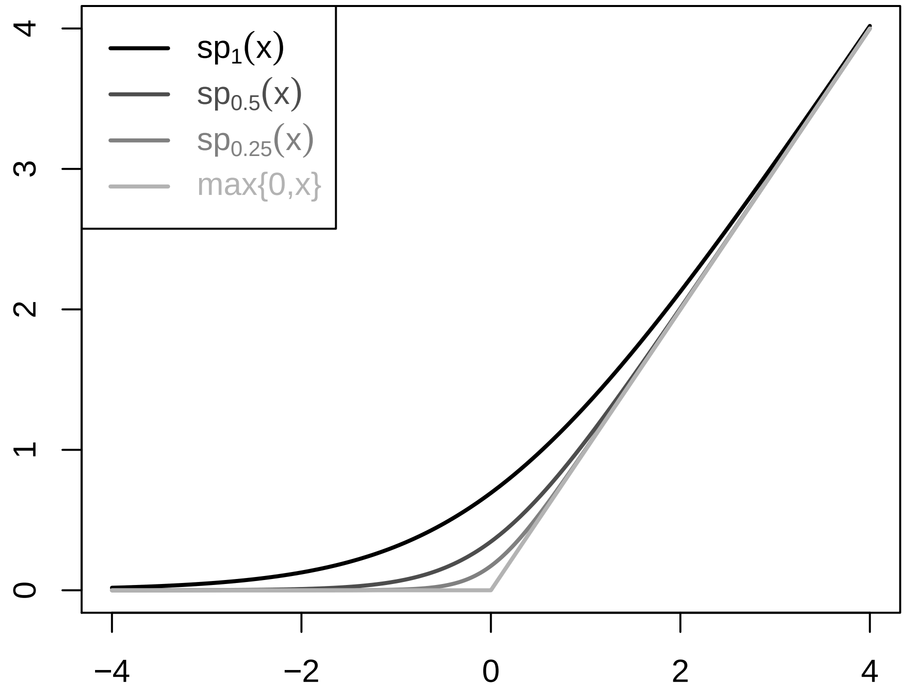
$\mathrm{sp}_c(x) = c\,\ln\big(1 + \exp(x/c)\big)$

with tuning parameter $c > 0$,

truly positive

and differentiable;

spINGARCH model:



$$M_t \;=\; s_c\Big(\alpha_0 \;+\; \sum_{i=1}^{p} \alpha_i\, X_{t-i} \;+\; \sum_{j=1}^{q} \beta_j\, M_{t-j}\Big).$$

Solution by Weiß & Jahn (2021) for bounded counts:

soft-clipping BINGARCH with response fct. $f = \mathrm{sc}_c$.

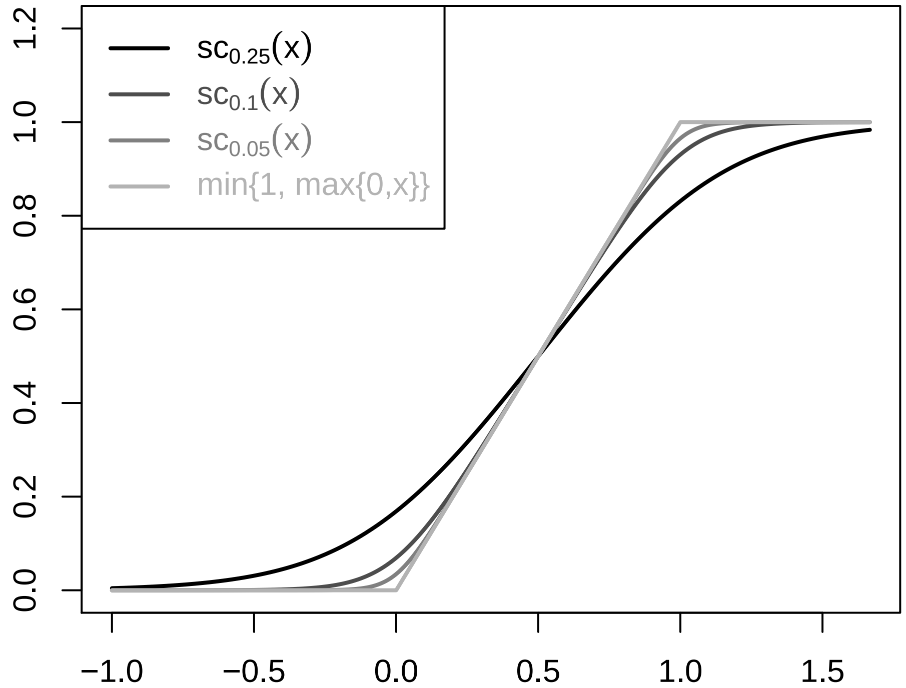
Soft-clipping function

$$\mathrm{sc}_c(x) = c \ln\left(\frac{1+\exp(\frac{x}{c})}{1+\exp(\frac{x-1}{c})}\right)$$

with tuning parameter $c > 0$,

truly in $(0;1)$

and differentiable;



scBINGARCH model:

$$P_t = \mathrm{sc}_c\Big(\alpha_0 + \sum_{i=1}^{p} \alpha_i\, X_{t-i}/n + \sum_{j=1}^{q} \beta_j\, P_{t-j}\Big).$$

On Linear Spatio-Temporal INGARCH Models

Existing Literature

Up to now, spatio-temporal (ST) models for *un*bounded counts, mainly applications in epidemiology (disease surveillance).

Let $\boldsymbol{X}_t = (X_{t,1}, \ldots, X_{t,m})^\top \in \mathbb{N}_0^m$, $m \in \mathbb{N}$, express count $X_{t,i}$ at time t in unit i. For INGARCH-type models, denote conditional mean $M_{t,i}$, used for generating $X_{t,i}|\boldsymbol{\mathcal{F}}_{t-1}$.

ST-INARCH(1) model (Held et al., 2005; Paul et al., 2008):

$$M_{t,i} \;=\; \lambda_i\, X_{t-1,i} \;+\; \phi_i \sum_{j\neq i} w_{ji}\, X_{t-1,j} \;+\; \gamma_i,$$

where, e. g., spatial weights $w_{ji} = 0$ if unit j not neighbour of i, otherwise $w_{ji} = 1/n_j$ (row-normalized, n_j=neighbours of j).

ST-INARCH(1) model in matrix notation: $\boldsymbol{M}_t = \boldsymbol{\Gamma}\,\boldsymbol{X}_{t-1} + \boldsymbol{\gamma}$,

where $\boldsymbol{\Gamma}$ diagonal entries λ_i and off-diagonal entries $\phi_i\, w_{ji}$,

see Held & Paul (2012).

Armillotta & Fokianos (2021): unique λ, ϕ, γ across locations i.

Higher-order extension by Bracher & Held (2020):

$$\boldsymbol{M}_t = \sum_{k=1}^{p} u_k\, \boldsymbol{\Gamma}\, \boldsymbol{X}_{t-k} + \boldsymbol{\gamma},$$

where AR-weights $u_1, \ldots, u_p$ with $\sum_{k=1}^{p} u_k = 1$, e. g.,

$u_k \sim \kappa^{k-1}$ such that INGARCH$(1,1)$-type model for $p \to \infty$.

ST-INGARCH$(1,1)$ by Clark & Dixon (2021):

$\boldsymbol{M}_t = \gamma_t + a_1\, \boldsymbol{X}_{t-1} + b_1\, \boldsymbol{M}_{t-1}$, i. e., with scalar coefficients.

Problems:

- All model parameters non-negative (to ensure $M_{t,i} > 0$), thus only positive ACF values can be captured.
- No proposal for bounded counts.

Our contributions:

- ST softplus INGARCH$(p, r; q, s)$ (ST-spINGARCH) for unbounded counts − skipped because of time restrictions;
- ST soft-clipping BINGARCH$(p, r; q, s)$ (ST-scBINGARCH) for bounded counts, applied to meteorological time series, solving problems we are confronted with in practice.

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The Spatio-Temporal Soft-clipping BINGARCH Model for Bounded Counts

Definition & Discussion

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ST-scBINGARCH$(p, r; q, s)$ model with range $\{0, \ldots, n\}^m$:

$$P_{t,i} \;=\; \text{sc}_c\Big(\alpha_0 + \sum_{k=1}^{p} \alpha_k\, X_{t-k,i}/n + \sum_{g=1}^{r} \lambda_g \textstyle\sum_{j\neq i} w_{ij}\, X_{t-g,j}/n$$
$$+ \sum_{l=1}^{q} \beta_l\, P_{t-l,i} + \sum_{h=1}^{s} \phi_h \textstyle\sum_{j\neq i} w_{ij}\, P_{t-h,j}\Big),\quad i = 1, \ldots, m,$$

where $\boldsymbol{X}_t | \boldsymbol{\mathcal{F}}_{t-1}$ emitted by

multivariate bounded-counts distribution with mean $n \cdot \boldsymbol{P}_t$.

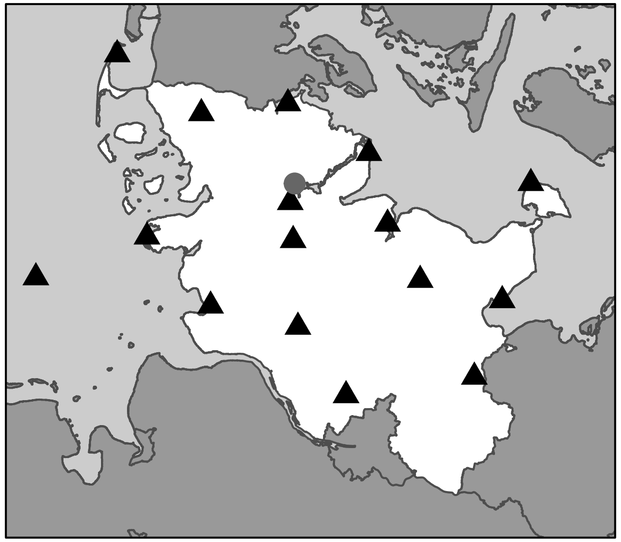
Default choice: (conditionally) independent binomials.

Possible extensions: copula for stronger cross-correlation, different distribution family, e. g., zero-inflated binomial (ZIB).

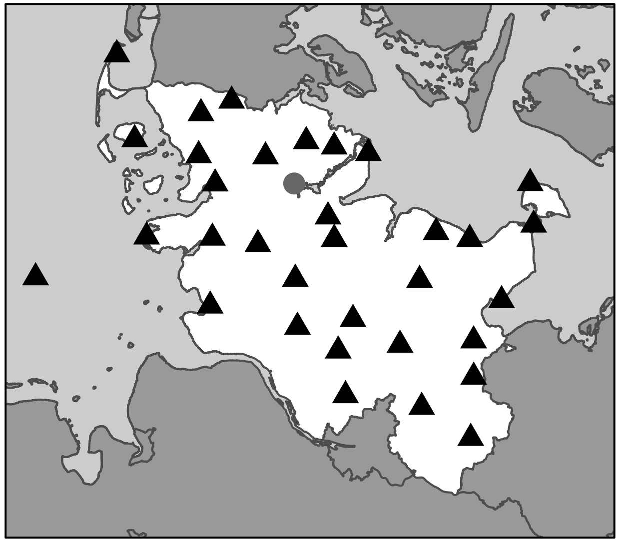
Parameter estimation by conditional ML approach.

Applied to meteorological t. s. (DWD Climate Data Center), for weather stations in Schleswig-Holstein:

cloud coverage data ($m = 17$):



precipitation data ($m = 34$):



grey circle = weather station "Schleswig"

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Cloud Coverage: Dealing with Missing Values

Data Application

Cloud Coverage: Dealing with Missing Values

Hourly cloud-coverage data (2009–2019), i. e.,

share of visible sky covered by clouds.

Counts (“okta”) from 0 (no clouds) to 8 (sky fully overcast),

i. e., counts $X_{t,i}$ with bound $n = 8$.

Irregular observations:

flag “9” (no data), “-1” (sky not observable),

treated as “NA” (not available, i. e., missing data).

5.5 % of NA observations in $\{x_{t,i}\}$. But

57.2 % of vectors $\boldsymbol{x}_t$ with at least one NA component $x_{t,i}$.

$\Rightarrow$ We have to **impute integer values**!

Cloud Coverage: Dealing with Missing Values

Our solution: If $x_{t,i}$ missing, then impute with value $x_{t,j}$ from most correlated non-NA station at time t (thus preserving integer nature of data).

Selected $x_{t,j} \approx$ "nearest neighbour" in terms of cross-correlation.

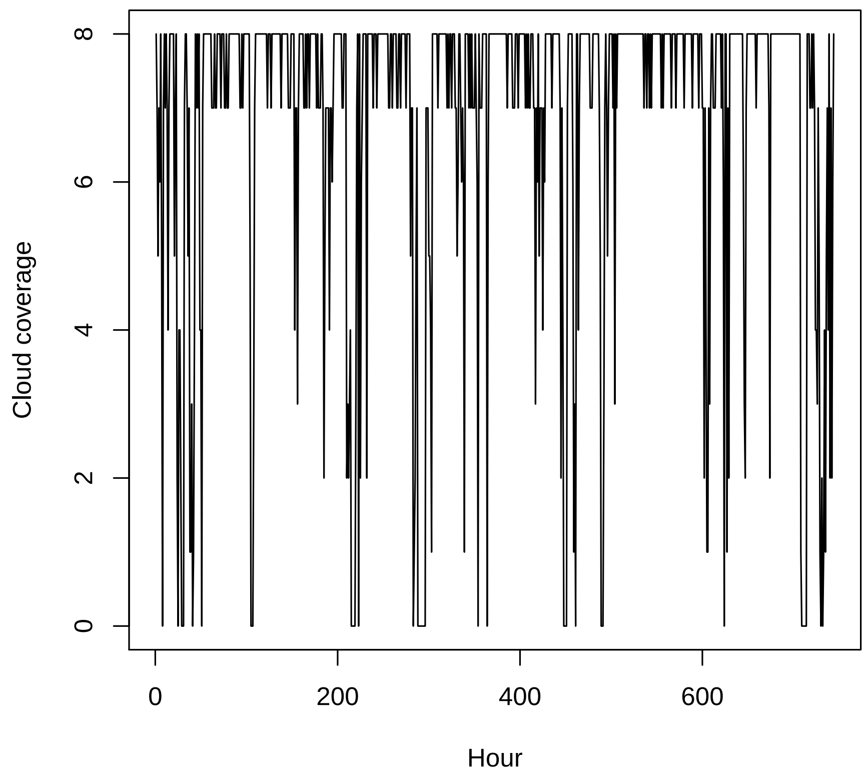
Average maximum corr. between stations over time is 0.709.

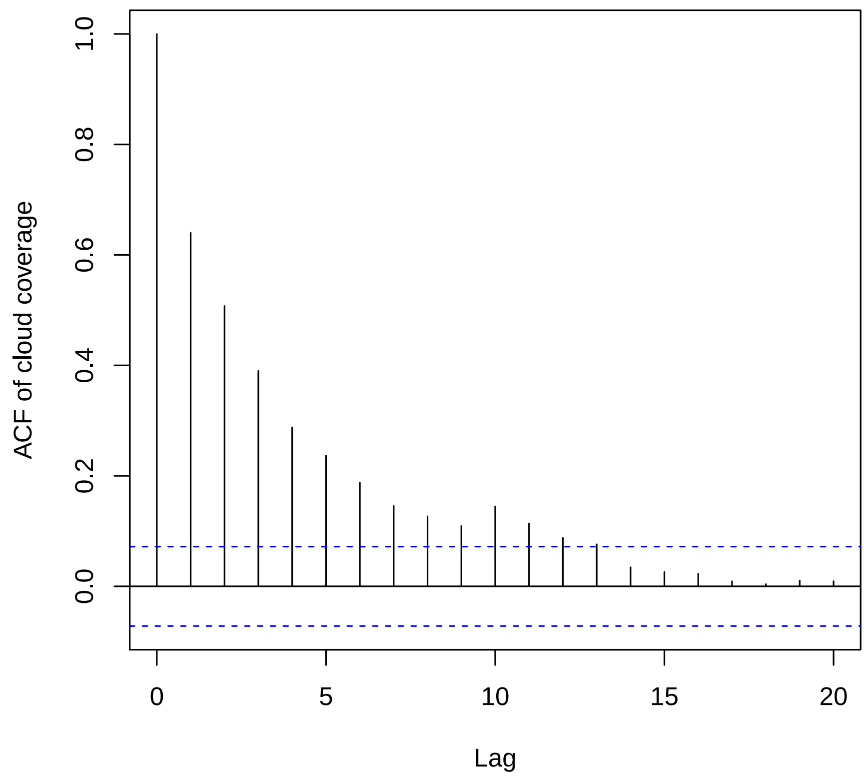
Our proposal for **spatial weight matrix W** $= (w_{ij})_{i,j=1,\ldots,m}$**:**
rely on great circle distance d_{ij}, with two extreme cases:

- sparse $\mathbf{W}^{(1)}$: only nearest neighbour,
  ${w_{ij}}^{(1)} = 1$ if j geographically closest to i;
- dense $\mathbf{W}^{(2)}$: all neighbours,
  ${w_{ij}}^{(2)} = 1/d_{ij}$ plus row-normalization.

Cloud Coverage: Dealing with Missing Values

Example: Hourly cloud coverage at "Schleswig", Dec. 2019:





$\Rightarrow$ slowly decaying memory,

use feedback terms in ST-scBINGARCH model.

For both $\mathbf{W}^{(1)}$ and $\mathbf{W}^{(2)}$, model selection via AIC.

$\mathbf{W}^{(2)}$ generally produces lower AIC values.

Cloud-coverage counts (Dec. 2019):

ML-fitted ST-scBINGARCH$(1,3;1,1)$ model with $\mathbf{W} = \mathbf{W}^{(2)}$.

	α_0	α_1	λ_1	λ_2	λ_3	β_1	ϕ_1
est. coeff.	0.0725	0.3496	0.6544	0.0909	-0.0637	0.2284	-0.3474
approx. SE	0.0060	0.0070	0.0124	0.0511	0.0089	0.0152	0.0515

Note: negative coefficients, soft-clipping link relevant.

Feedback terms β_1 and ϕ_1 have opposing signs:

"compensation effect", for ACF of "foreign" stations,

we have faster decay than under pure AR structure.

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Rainy Hours per Day: Accounting for Zero-Inflation

Data Application

Rainy Hours: Accounting for Zero-Inflation

Rainy hours per day: $x_{t,i}$ with upper limit $n = 24$.

Again missing data (0.36 % of total hourly observations),

but in particular, strong **zero inflation** (38 % of zeros).

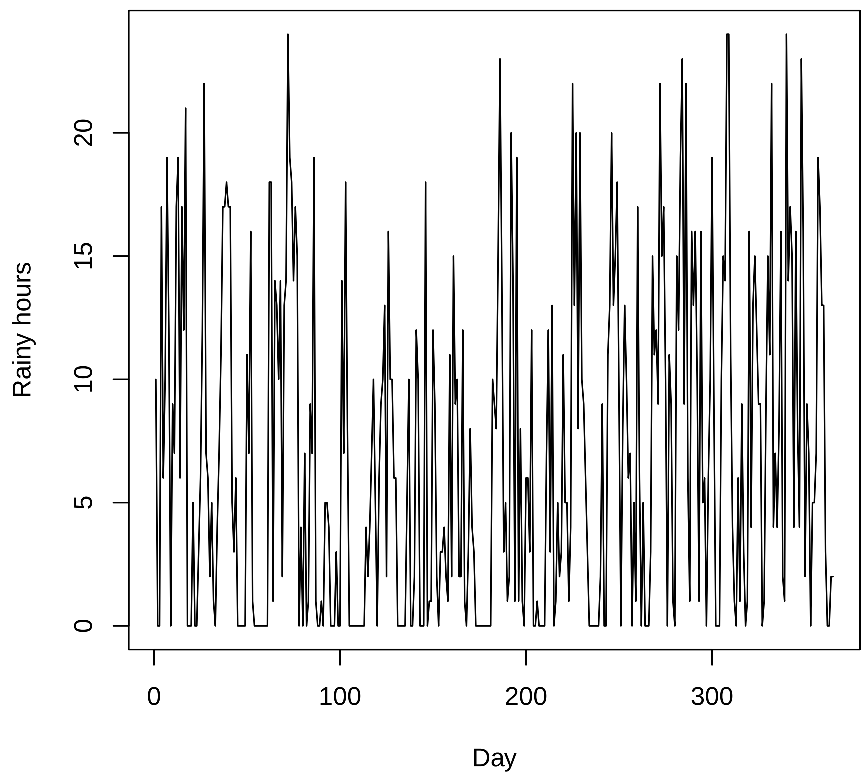
Thus, **conditional ZIB distribution** with parametrization

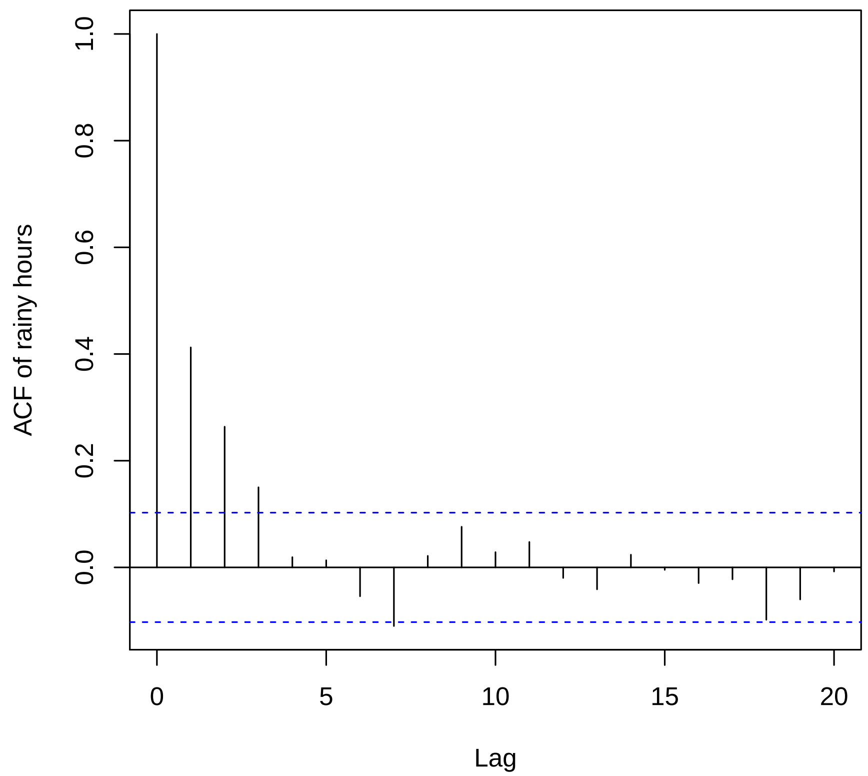
$$P\big(X_{t,i} = x|\mathcal{F}_{t-1}\big) \; = \; \omega\,(1 - P_{t,i})\,\mathbb{1}_{\{x=0\}} + \big(1 - \omega\,(1 - P_{t,i})\big)\,p_{\text{Bin}}\big(x\,|\,n, \tfrac{P_{t,i}}{1-\omega\,(1-P_{t,i})}\big).$$

(. . .) see plots on next slide!

$\mathbf{W} = \mathbf{W}^{(2)}$	ω	α_0	α_1	α_2	α_3	λ_1	λ_2	λ_3
est. coeff.	0.4359	0.1370	0.1863	0.0986	0.1033	0.0336	-0.0396	-0.1057
approx. SE	0.0047	0.0016	0.0074	0.0082	0.0075	0.0086	0.0096	0.0087

Example: Rainy hours per day at "Schleswig" in 2019:





$\Rightarrow$ ACF drops quickly towards zero,

feedback-terms not needed this time.

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Cloud Coverage: Accounting for Cross-Correlation

Data Application

Cloud Coverage: Accounting for Cross-Correlation

Default model (conditionally independent binomials)

only moderate cross-correlation.

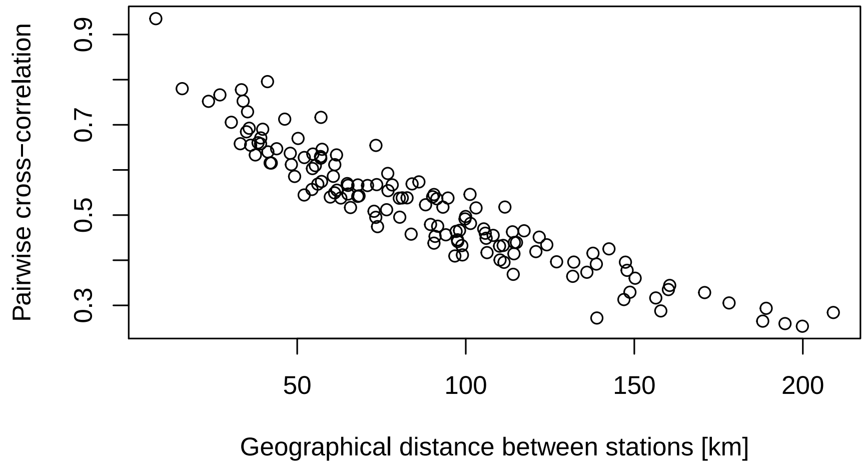
Cloud-coverages (Dec. 2019) with **strong cross-correlation**:

average cross-corr. ≈ 0.52,

negative relationship
to geographical distance.

Default model:

cross-correlation ≈ 0.25.



Idea: like in Armillotta & Fokianos (2021),

employ **copula** distributions with count marginals.

Gaussian copula

$$C(u_1,\ldots,u_m) \;=\; \Phi_{m;\mathbf{R}}\big(\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_m)\big)$$

with corresponding copula density

$$c(u_1,\ldots,u_m) \;=\; \det(\mathbf{R})^{-1/2}\,\exp\Big(\tfrac{1}{2}\,\Phi^{-1}(\boldsymbol{u})^\top\,\big(\mathbf{I}-\mathbf{R}^{-1}\big)\,\Phi^{-1}(\boldsymbol{u})\Big).$$

With univariate count CDFs $F_1,\ldots,F_m$, joint multivariate CDF

$$F(x_1,\ldots,x_m) \;=\; C\big(F_1(x_1),\ldots,F_m(x_m)\big).$$

Problem: joint PMF $p(x_1,\ldots,x_m)$ requires 2^m discrete differences of joint CDF, not feasible!

Gaussian copula (. . .)

We use approximate solution of Kazianka & Pilz (2010):

$$p(x_1, \ldots, x_m) \approx c\big(u_1, \ldots, u_m\big) \cdot \prod_{i=1}^{m} P(X_i = x_i),$$

where $u_i = \frac{1}{2}\big(F_i(x_i) + F_i(x_i - 1)\big)$ ("mid-point approximation").

How to choose **R** for $\Phi_{m;\mathbf{R}}(\cdot)$?

Standard solution:

"exchangeable correlation structure", $\mathbf{R} = (1-\rho)\,\mathbf{I} + \rho\,\mathbf{E}$.

Cloud Coverage: Accounting for Cross-Correlation

How to choose $\mathbf{R}$ for $\Phi_{m;\mathbf{R}}(\cdot)$? $(\ldots)$

To explicitly account for spatial pattern in cross-correlation, we resort to concept of spatially correlated errors.

Spatial error model (SEM): $\boldsymbol{y} = f(\boldsymbol{x}, \boldsymbol{\beta}) + \boldsymbol{u}$, where $\boldsymbol{u} = \rho\,\mathbf{W}\boldsymbol{u} + \boldsymbol{\epsilon}$ with standard normal $\boldsymbol{\epsilon}$ (LeSage & Pace, 2009).

Error $\boldsymbol{u}$ as $\boldsymbol{u} = (\mathbf{I} - \rho\,\mathbf{W})^{-1}\,\boldsymbol{\epsilon}$.
With $\mathbf{B} = (\mathbf{I} - \rho\,\mathbf{W})^{-1}$, resulting covariance matrix $\mathbf{B}\mathbf{B}^\top$.

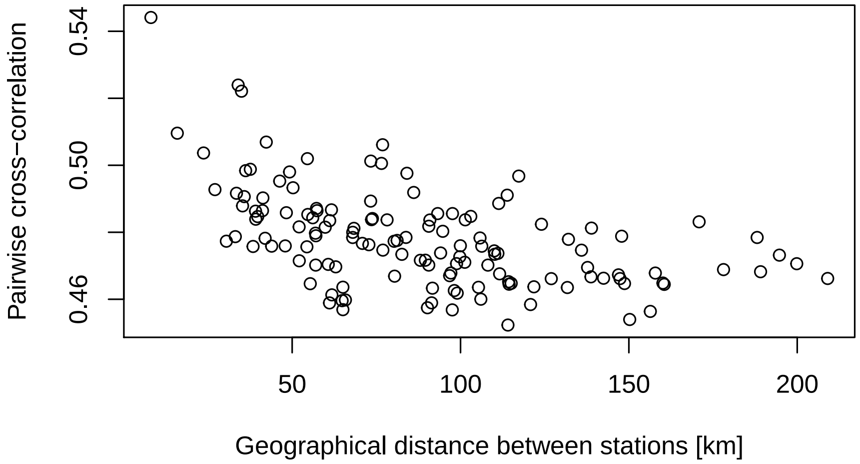
Idea: "SEM copula" with covariance matrix $\mathbf{B}\mathbf{B}^\top$, again single dependence parameter ρ.

SEM-copula ST-scBINGARCH$(1,3;1,1)$ with $\mathbf{W} = \mathbf{W}^{(2)}$:

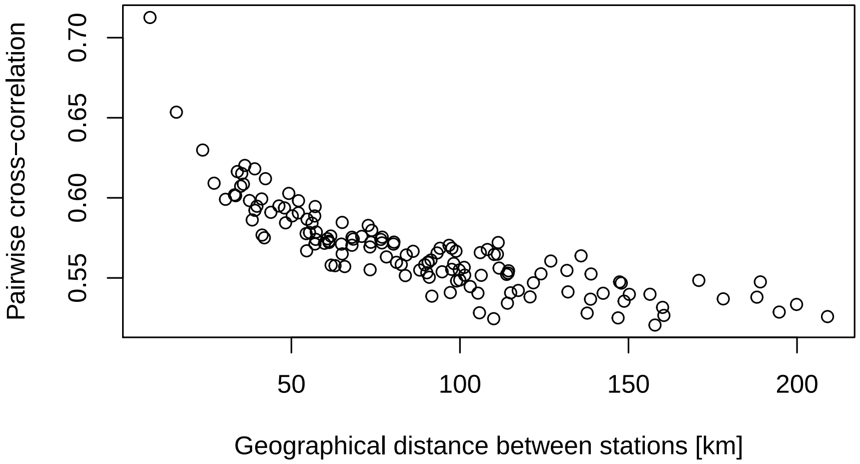
	ρ	α_0	α_1	λ_1	λ_2	λ_3	β_1	ϕ_1
est. coeff.	0.4149	0.0566	0.3452	0.6567	0.0644	-0.0496	0.2335	-0.3181
approx. SE	0.0010	0.0105	0.0079	0.0195	0.0242	0.0211	0.0094	0.0276

Relation between correlation and distance for

standard copula:



SEM copula:



Thank You

for Your Interest!

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