SPC Methods for Time-Dependent Processes of Counts

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Introduction
In many fields of application, we are concerned with count data processes. Typical examples are counts of defects produced per item in manufacturing industry, counts of new cases of an infection per time unit in health care monitoring, or counts of complaints by customers per time unit in service industry. Often, it is important to detect changes in the process as soon as possible to be able to start preventive actions or to avoid further damages. Methods of statistical process control (SPC) are a suitable tool for this purpose.

During the last years, there was increasing interest in SPC methods for time-dependent processes of counts. In the sequel, feasible models for autocorrelated counts processes are presented, approaches for corresponding control charts are considered, and process capability indices are discussed. More details and full references can be found in the open access article Weiß CH (2015) SPC methods for time-dependent processes of counts—a literature review. Cogent Mathematics, 2, 1111116.

Basic models for autocorrelated counts processes
One of the earliest approaches toward stationary count data processes is the INAR(1) model by [M11], defined by \( X_t = X_{t-1} + Y_t \), where the innovations \( Y_t \) are i.i.d. counts, and where \( \ast \) denotes the binomial thinning operator.

Several modifications to the basic INAR(1) model have been proposed, e.g., where the binomial thinning operator is replaced by another type of thinning [SW15]. As an example, [BR09] introduced the NGINAR(1) model, defined by \( X_t = X_{t-1} + Y_t \), where \( \ast \) denotes the negative binomial thinning operator.

Another popular approach is the INARCH(1) model, where \( X_t \) is conditionally Poisson distributed in the following way:
\[
X_t | X_{t-1}, \ldots \sim \text{Poi}(\alpha X_{t-1} + \gamma).
\]

If systematic trend and seasonality have to be incorporated, regression models like the autoregressive log-linear model (ARLM) can be used, with time-dependent mean \( \mu_t \) given by
\[
\ln(\mu_t) = \theta_0 + \theta_1 t + \sum_{s=1}^S \{ \theta_{s+2} \cos(2\pi s t) + \theta_{s+3} \sin(2\pi s t) \}.
\]

Common SPC methods
Control charts
A basic approach is to plot the counts \( X_t \) directly on a chart with appropriately chosen control limits \( \bar{c} \) chart (if counts have full range \( 0 \) or up to 39.3 \( \bar{c} \)) or Shewhart-type chart (if counts range \( 0 \) to \( n \)). Applications of the \( \bar{c} \) chart to INAR(1) processes were considered by [W07, W11b], to consider autocorrelation (not even the basic “actual proportion of conformity”). If the upper specification limit USL describes, e.g., the maximal acceptable number of non-conformities per produced item, then the probability \( P(X > USL) \) is compared to a prespecified acceptable probability level 1 - \( \alpha \). [P05] used an upper-sided CUSUM chart at time \( t \) to ask how to compute the ARLs for approximated based on simulations with a sufficiently high sample size. In the context of autocorrelated count data processes, this topic was considered by, e.g., [ZNHH14, WT15] for the Poisson INAR(1) model and diverse types of control charts.

Another popular approach is the EWMA chart, which is defined by
\[
C_t = \alpha X_t + (1 - \alpha) C_{t-1}.
\]

If \( \alpha = 0 \) or 1, the resulting chart, respectively, is defined by
\[
C_t = \begin{cases} X_t & \text{if } \alpha = 0, \\ \text{mean} & \text{if } \alpha = 1. \end{cases}
\]

In the context of monitoring autocorrelated counts processes, the upper-sided \( CUSUM \) was applied to INAR(1) processes by [W07, W11b], to INARCH(1) processes by [WT12], the lower-sided and the two-sided version were applied to INAR(n) processes by [WYTB13]. For performance evaluation, it is important that the \( CUSUM \) preserves the discrete nature of the process. Therefore, exact run length computations are possible with a type of MC approach [W11b]: the one-sided \( CUSUM \) requires to estimate the process mean, the \( CUSUM \) is defined by
\[
C_t = \alpha X_t + (1 - \alpha) C_{t-1}.
\]

If \( \alpha = 0 \) or 1, the resulting chart, respectively, is defined by
\[
C_t = \begin{cases} X_t & \text{if } \alpha = 0, \\ \text{mean} & \text{if } \alpha = 1. \end{cases}
\]

Process capability indices
Only few of the works about capability indices refer to attributes data processes. [P05] picked up the idea of considering the actual “proportion of conformance”. If the upper specification limit USL describes, e.g., the maximal acceptable number of non-conformities per produced item, then the probability \( P(X > USL) \) is compared to a prespecified acceptable probability level 1 - \( \alpha \). [P05] considered this task for an underlying Poisson counts process, \( W12 \) extended this work to an underlying Poisson INAR(1) process, distinguishing between the process capability for observations or innovations, respectively.

Advanced control charts
The basic chart allows for a continuous monitoring of a serially dependent count data process, but the statistic plotted on the chart at time \( t \) (+1st observation) does not comprise information about past values of the process (not beyond the mere effect of autocorrelation). Therefore, it is not necessarily sensitive to small or moderate changes in the process. For this reason, several types of advanced control charts have been proposed, where the statistic plotted at time \( t \) also uses past observations of the process and hence accumulates information about the process for a longer period of time.

CUSUM charts
The traditional cumulative sum (CUSUM) control chart \( [P54] \) applied directly to the observations \( X_t \) is perhaps the most natural advanced candidate for monitoring autocorrelated processes of counts, because it preserves the discrete nature of the process by only using additions (but no multiplications). The upper-sided CUSUM is defined by
\[
C_t^+ = \max(0, X_t - \gamma + C_{t-1}^+), \quad C_{t-1}^+ \geq 0,
\]

where an alarm is triggered if \( C_t^+ \geq \gamma \) (control limit). While the upper-sided CUSUM is mainly designed to detect increases in the process mean, the lower-sided CUSUM aims at uncovering decreases in the process mean.

If \( (C_t^-, C_t^+) \) are monitored simultaneously, then this chart combination is referred to as a two-sided CUSUM chart:
\[
C_t^\pm = \max(0, X_t - \gamma + C_{t-1}^\pm), \quad C_{t-1}^\pm \geq 0.
\]

New CUSUM charts are developed for autocorrelated count data processes. Therefore, the CUSUM chart (any Shewhart-type chart) is only sensitive to small mean shifts and rather insensitive to process changes. Therefore, [W11b] proposed a modification, where a refined rounding operation is used: for \( \alpha \), the round \( \alpha \) to the nearest fraction with denominator \( \alpha \) can be computed again exactly by adapting the MC approach.

References
[BS11] Brereton TP, Evans DA (2011) A detailed description together with corresponding software implementations is provided by [W07, W11b].