The INARCH(1) Model for Overdispersed Time Series of Counts

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Some introductory words . . .

This talk is based on the articles

Weiß, C.H. (2009a). *Modelling time series of counts with overdispersion*. Appears in Statistical Methods and Applications.

Weiß, C.H. (2009b). *The INARCH(1) Model for Overdispersed Time Series of Counts*. Submitted.

All references mentioned in this talk correspond to the references in these articles.

Processes of Counts with Overdispersion

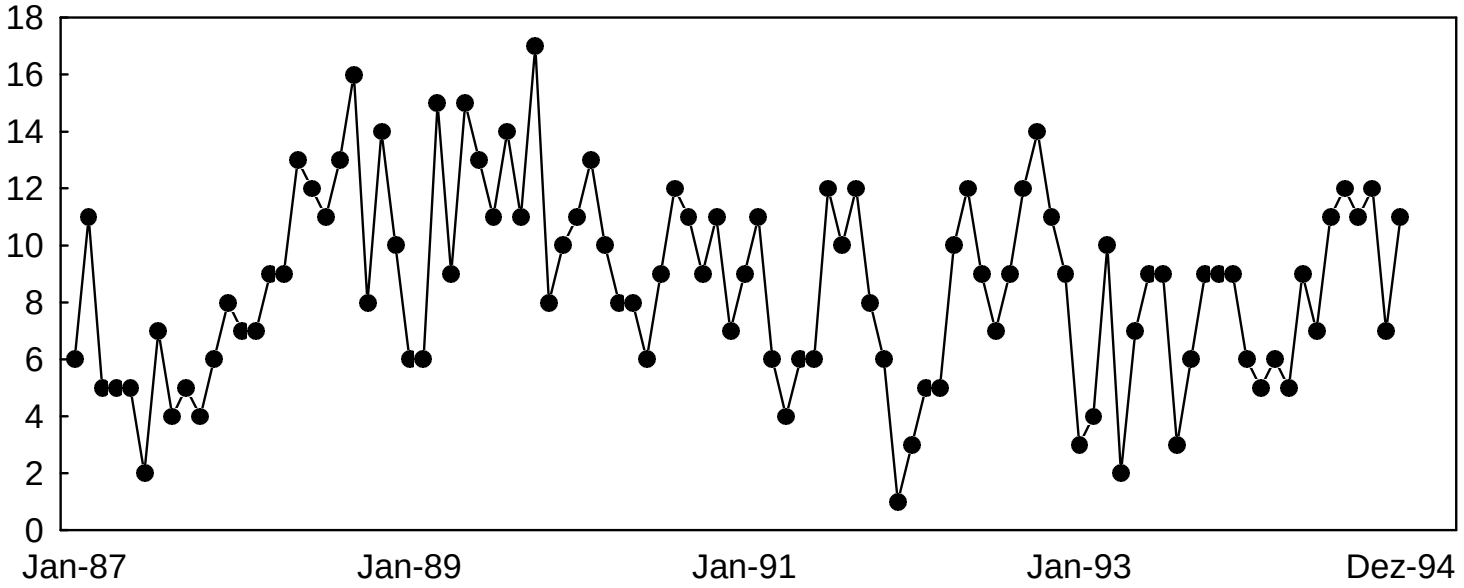
Motivation

Counts with Overdispersion

Processes of counts commonly observed in real-world applications. Examples from diverse fields in practice:

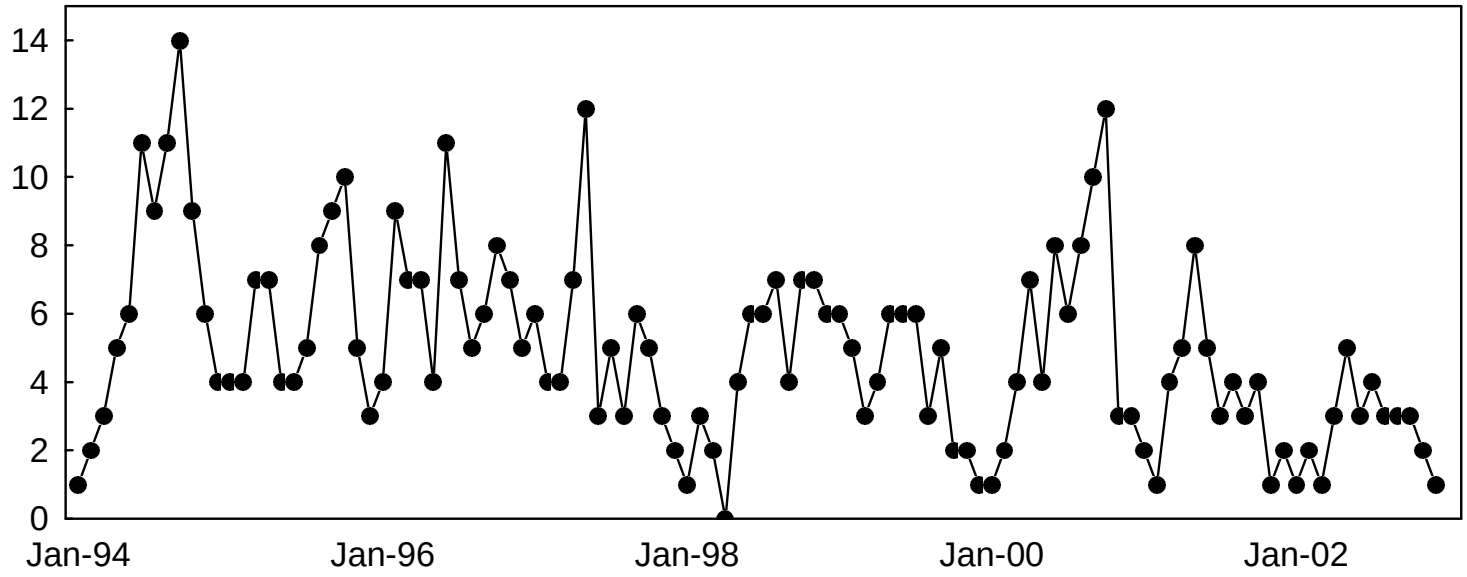
- insurance (e. g., time series of claim counts),
- economics (e. g., counts of price changes),
- statistical process control (e. g., counts of defects),
- traffic (e. g., counts of accidents),
- network monitoring (e. g., intrusion detection system),
- epidemiology (e. g., counts of diseases), and others.

Example 1: Monthly claims counts (1987 to 1994): burn related injuries in heavy manufacturing industry. Source: Freeland (1998).



Example 2: Monthly strike data (1994 to 2002): number of work stoppages leading to 1000 workers or more being idle in effect in the period.

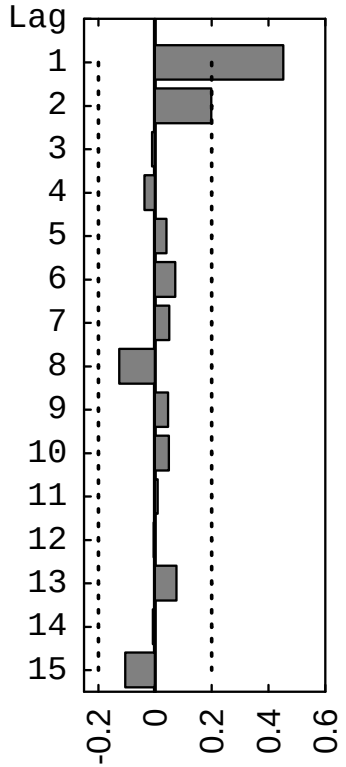
Source: U.S. Bureau of Labor Statistics.



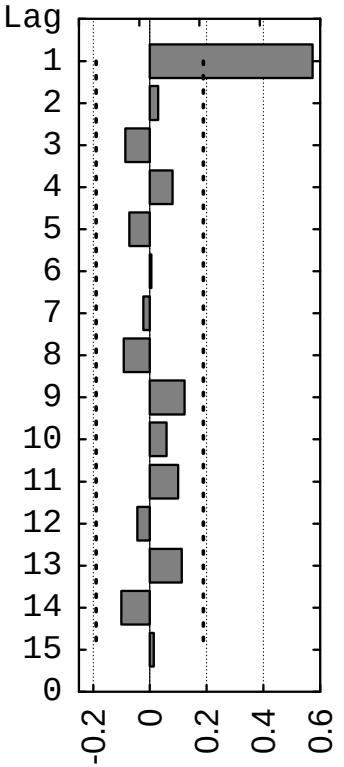
Analysis of both time series:

Partial autocorrelation function of

Example 1:



Example 2:



Analysis of both time series: (continued)

For both examples, AR(1)-like dependence structure

$\Rightarrow$ Popular Poisson INAR(1) model appropriate?

Analysis of both time series: (continued)

Marginal properties:

- Example 1: mean 8.60 (SE$\approx$ 0.49) and variance 11.36;
- Example 2: mean 4.94 (SE$\approx$ 0.41) and variance 7.92.

$\Rightarrow$ **Overdispersion** for both examples!

$\Rightarrow$ The popular Poisson INAR(1) model cannot be used!

Counts with Overdispersion

Overdispersion commonly observed in practice.

Typical reasons:

- presence of positive correlation between monitored events (Friedman, 1993; Poortema, 1999; Paroli et al., 2000);
- variation in probability of monitored events (Heimann, 1996; Poortema, 1999; Christensen et al., 2003);
- further potential causes of overdispersion discussed by Jackson (1972).

In a nutshell:

Modeling of real-world time series of counts

usually requires to consider both

serial dependence and overdispersion!

Recent approach:

INGARCH models, the *in*teger-valued *g*eneralized *a*uto-*r*egressive *c*onditional *h*eteroskedasticity models.

The INGARCH Models

Definition & Properties

INGARCH(p, q) model introduced by Heinen (2003):

$(X_t)_{\mathbb{Z}}$ with range $\mathbb{N}_0$ follows **INGARCH(p, q) model** with $p \geq 1$ and $q \geq 0$ if

(i) X_t, conditioned on $X_{t-1}, \ldots$, is Poisson-distributed according to $Po(M_t)$, where

(ii) conditional mean $M_t := E[X_t \mid X_{t-1}, \ldots]$ fulfills

$$M_t = \beta_0 + \sum_{i=1}^{p} \alpha_i \cdot X_{t-i} + \sum_{j=1}^{q} \beta_j \cdot M_{t-j},$$

where $\beta_0 > 0$ and $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q \geq 0$.

Condition (i) $\Rightarrow$

conditional distribution of $X_t\big|_{X_{t-1},\ldots}$ **equi**dispersed, i. e.,

$E[X_t \mid X_{t-1},\ldots] = M_t = V[X_t \mid X_{t-1},\ldots]$.

But **unconditional** distribution shows **over**dispersion:

$$\mu_t := E[X_t] = E[M_t],$$

$$V[X_t] = E[M_t] + V[M_t] = \mu_t + V[M_t].$$

$\Rightarrow$ INGARCH(p, q) model useful for

modeling time series of overdispersed counts!

Ferland et al. (2006):

For $\alpha_\bullet + \beta_\bullet := \sum_{i=1}^{\mathrm{p}} \alpha_i + \sum_{j=1}^{\mathrm{q}} \beta_j < 1$,

the INGARCH(p, q) model exists and is strictly stationary,

with finite first and second order moments.

Mean $\mu := E[X_t] = \beta_0/(1 - \alpha_\bullet - \beta_\bullet)$.

Weiß (2009a):

Yule-Walker-type equations for $\gamma_X(k) := Cov[X_t, X_{t-k}]$:

$$V[X_t] = \mu + \sum_{i=1}^{\mathsf{p}} \alpha_i \cdot \gamma_X(i) + \sum_{j=1}^{\mathsf{q}} \beta_j \cdot \gamma_M(j),$$

$$\gamma_X(k) = \sum_{i=1}^{\mathsf{p}} \alpha_i \cdot \gamma_X\big(|k-i|\big) + \sum_{j=1}^{\min\{k-1,\mathsf{q}\}} \beta_j \cdot \gamma_X(k-j) + \sum_{j=k}^{\mathsf{q}} \beta_j \cdot \gamma_M(j-k) \quad \text{for } k \geq 1,$$

where $\gamma_M(k) := Cov[M_t, M_{t-k}]$ with

$$\gamma_M(k) = \sum_{i=1}^{\min\{k,\mathsf{p}\}} \alpha_i \cdot \gamma_M(k-i) + \sum_{i=k+1}^{\mathsf{p}} \alpha_i \cdot \gamma_X(i-k) + \sum_{j=1}^{\mathsf{q}} \beta_j \cdot \gamma_M\big(|k-j|\big) \quad \text{for } k \geq 0.$$

Further simplification for q = 0, i. e.,

for INARCH(p) family with $\alpha_\bullet < 1$:

$$\gamma_X(k) \;=\; \sum_{i=1}^{\mathsf{p}} \alpha_i \cdot \gamma_X(|k-i|) \;+\; \delta_{k0} \cdot \mu \qquad \text{for } k \geq 0.$$

Nearly identical to YW equations of standard AR(p) models.

Consequence:

Model order p can be identified

via usual partial autocorrelation function $\rho_{\mathsf{part}}(k)$:

$\rho_{\mathsf{part}}(k) = 0$ for $k > \mathsf{p}$.

In the following,
we concentrate on most simple INARCH model,
the **INARCH(1) model**.

As seen above,
INARCH(1) model has AR(1)-like dependence structure
(like in above examples), i. e.,
INARCH(1) model is alternative to
popular Poisson INAR(1) model,
but is able to model overdispersion.

The INARCH(1) Model

Definition & Properties

Definition:

Let $(X_t)_{\mathbb{Z}}$ be a process with range $\mathbb{N}_0 = \{0, 1, \ldots\}$,

let $\beta > 0$ and $0 < \alpha < 1$.

$(X_t)_{\mathbb{Z}}$ is said to follow an **INARCH(1) model**

if X_t, conditioned on $X_{t-1}, X_{t-2}, \ldots$,

is Poisson distributed according to $Po(\beta + \alpha \cdot X_{t-1})$.

Basic properties:

- Stationary Markov chain with transition probabilities

$$\begin{aligned} p_{i|j} &:= P(X_t = i \mid X_{t-1} = j) \\ &= \exp(-\beta - \alpha \cdot j) \cdot \frac{(\beta + \alpha \cdot j)^i}{i!} > 0; \end{aligned}$$

- autocorrelation function $\rho_X(n) := Corr[X_t, X_{t-n}] = \alpha^n$.

Weiß (2009a): (Marginal Cumulants)

The cumulants follow recursively from

$$\kappa_1 = \frac{\beta}{1-\alpha}, \qquad \kappa_n = -(1-\alpha^n)^{-1} \cdot \sum_{j=1}^{n-1} s_{n,j} \cdot \kappa_j \quad \text{for } n \geq 2,$$

where $s_{n,j}$ are Stirling numbers of first kind:

$$s_{n,0} = 0 \quad \text{and} \qquad s_{n,n} = 1 \quad \text{for } n \geq 1,$$
$$s_{n+1,j} = s_{n,j-1} - n \cdot s_{n,j} \qquad \text{for } j = 1, \ldots, n \text{ and } n \geq 1.$$

Weiß (2009a): (Marginal Cumulants) (continued)

In particular,

$$\kappa_1 = \frac{\beta}{1-\alpha} = E[X_t], \quad \kappa_2 = \frac{\beta}{(1-\alpha)(1-\alpha^2)} = V[X_t],$$

i. e., **overdispersion**,

$$\kappa_3 = \frac{1+2\alpha^2}{1-\alpha^3} \cdot \kappa_2, \quad \kappa_4 = \frac{1+6\alpha^2+5\alpha^3+6\alpha^5}{(1-\alpha^3)(1-\alpha^4)} \cdot \kappa_2,$$

i. e., skewness and excess of X_t are given by $\frac{1+2\alpha^2}{1+\alpha+\alpha^2} \cdot \sqrt{\frac{1+\alpha}{\beta}}$ and $\frac{1+6\alpha^2+5\alpha^3+6\alpha^5}{\beta(1+\alpha+\alpha^2)(1+\alpha^2)}$, respectively.

The INARCH(1) Model

Parameter Estimation

Conditional maximum likelihood approach:

$$L(\beta,\alpha) := P(X_T = x_T,\ldots,X_2 = x_2 \mid X_1 = x_1)$$

$$= \frac{e^{-(T-1)\beta}\cdot e^{-\alpha\sum_{t=2}^{T} x_{t-1}}\cdot\prod_{t=2}^{T}(\beta+\alpha x_{t-1})^{x_t}}{\left(\prod_{t=2}^{T} x_t!\right)}.$$

ML estimates via numerical maximization.

Observed Fisher information given by

$$\mathbf{J}(\beta,\alpha) = \begin{pmatrix} \sum_{t=2}^{T}\frac{x_t}{(\beta+\alpha x_{t-1})^2} & \sum_{t=2}^{T}\frac{x_t x_{t-1}}{(\beta+\alpha x_{t-1})^2} \\ \sum_{t=2}^{T}\frac{x_t x_{t-1}}{(\beta+\alpha x_{t-1})^2} & \sum_{t=2}^{T}\frac{x_t x_{t-1}^2}{(\beta+\alpha x_{t-1})^2} \end{pmatrix}.$$

$\rightarrow$ Approximate asymptotic SE of ML estimators.

INARCH(1) model performs very well for initial examples:

- **Example 1:**

  ML-estimates $\widehat{\beta} = 4.3796$ and $\widehat{\alpha} = 0.4911$,

  model mean 8.61 and variance 11.34,

  empirical mean 8.60 and variance 11.36.

- **Example 2:**

  ML-estimates $\widehat{\beta} = 1.8114$ and $\widehat{\alpha} = 0.6364$,

  model mean 4.98 and variance 8.37,

  empirical mean 4.94 and variance 7.92.

Conditional least squares approach:

$$\begin{aligned} CSS(\beta,\alpha) &:= \Sigma_{t=2}^{T} \left(x_t - E[X_t \mid X_{t-1} = x_{t-1}]\right)^2 \\ &= \Sigma_{t=2}^{T} \left(x_t - \beta - \alpha x_{t-1}\right)^2. \end{aligned}$$

Minimization leads to

$$\widehat{\alpha}_{\mathsf{CLS}} = \frac{\Sigma_{t=2}^{T} X_t X_{t-1} - \frac{1}{T-1} \cdot \Sigma_{t=2}^{T} X_t \cdot \Sigma_{s=2}^{T} X_{s-1}}{\Sigma_{t=2}^{T} X_{t-1}^2 - \frac{1}{T-1} \cdot \left(\Sigma_{t=2}^{T} X_{t-1}\right)^2},$$

$$\widehat{\beta}_{\mathsf{CLS}} = \frac{1}{T-1}\left(\Sigma_{t=2}^{T} X_t - \widehat{\alpha}_{\mathsf{CLS}} \cdot \Sigma_{t=2}^{T} X_{t-1}\right).$$

Weiß (2009b): (Asymptotic Distribution CLS)

Using a result of Klimko & Nelson (1978), it follows that

$$\sqrt{T-1}\,(\hat{\beta}_{\text{CLS}}-\beta,\hat{\alpha}_{\text{CLS}}-\alpha)^{\top} \xrightarrow{D} N(\mathbf{0},\boldsymbol{\Sigma}_{\beta,\alpha}),$$

where $\boldsymbol{\Sigma}_{\beta,\alpha} =$

$$\begin{pmatrix} \frac{\beta}{1-\alpha}\big(\beta(1+\alpha)+\frac{1+2\alpha^4}{1+\alpha+\alpha^2}\big) & -\beta(1+\alpha)-\frac{(1+2\alpha)\alpha^3}{1+\alpha+\alpha^2} \\ -\beta(1+\alpha)-\frac{(1+2\alpha)\alpha^3}{1+\alpha+\alpha^2} & (1-\alpha^2)\big(1+\frac{\alpha(1+2\alpha^2)}{\beta(1+\alpha+\alpha^2)}\big) \end{pmatrix}.$$

$\rightarrow$ Approximate asymptotic SE of CLS estimators, derivation of simultaneous confidence regions.

Weiß (2009b): (Simultaneous Confidence Regions)

Two types of confidence region based on CLS estimators:

$$\Big\{(\beta,\alpha) \mid (\hat{\beta}_{\text{CLS}}-\beta, \hat{\alpha}_{\text{CLS}}-\alpha)\boldsymbol{\Sigma}^{-1}_{\beta,\alpha}(\hat{\beta}_{\text{CLS}}-\beta, \hat{\alpha}_{\text{CLS}}-\alpha)^{\top} < \tfrac{z}{T-1}\Big\},$$

where z denotes γ-quantile of χ^2_2-distribution, and

$$\Big\{(\beta,\alpha) \mid$$

$$(\hat{\beta}_{\text{CLS}}-\beta)^2 < \tfrac{z^2}{T-1}\cdot\frac{\hat{\beta}_{\text{CLS}}\big(1+2\hat{\alpha}^4_{\text{CLS}}+\hat{\beta}_{\text{CLS}}(1+\hat{\alpha}_{\text{CLS}})(1+\hat{\alpha}_{\text{CLS}}+\hat{\alpha}^2_{\text{CLS}})\big)}{1-\hat{\alpha}^3_{\text{CLS}}},$$

$$(\hat{\alpha}_{\text{CLS}}-\alpha)^2 < \tfrac{z^2}{T-1}\cdot\big(1-\hat{\alpha}^2_{\text{CLS}}+\tfrac{\hat{\alpha}_{\text{CLS}}+\hat{\alpha}^3_{\text{CLS}}-2\hat{\alpha}^5_{\text{CLS}}}{\hat{\beta}_{\text{CLS}}(1+\hat{\alpha}_{\text{CLS}}+\hat{\alpha}^2_{\text{CLS}})}\big)\Big\},$$

where z denotes $(3+\gamma)/4$-quantile of $N(0,1)$-distribution.

Performance for finite values of T via simulation study using Mathematica 5, with 50,000 replications per (β, α), γ, T.

Main results:

For $\alpha \leq 0.2$, CLS_1 region reliable already if $T \geq 100$.

For $0.4 \leq \alpha \leq 0.6$, CLS_1 region requires $T \geq 500$,

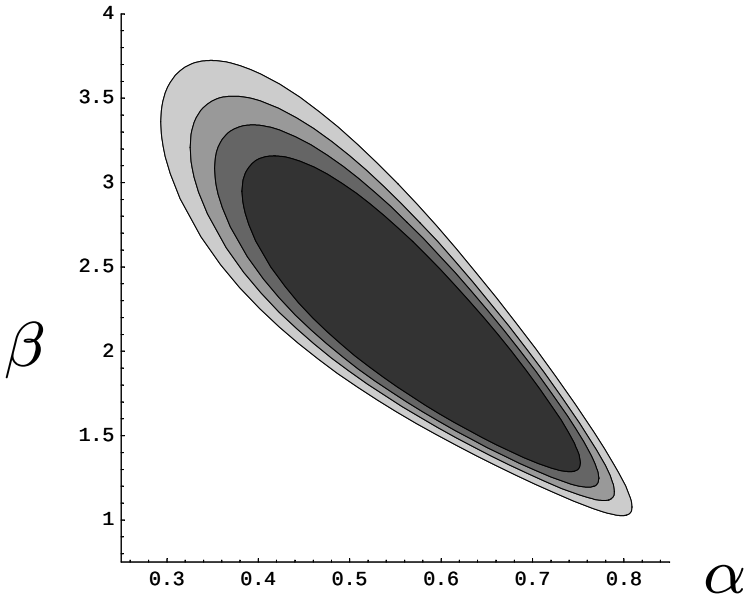
while $\alpha \approx 0.8$ even requires $T \geq 1000$.

CLS_2 region conservative, but can be used if $\gamma \geq 0.95$ independent of (β, α) and T.

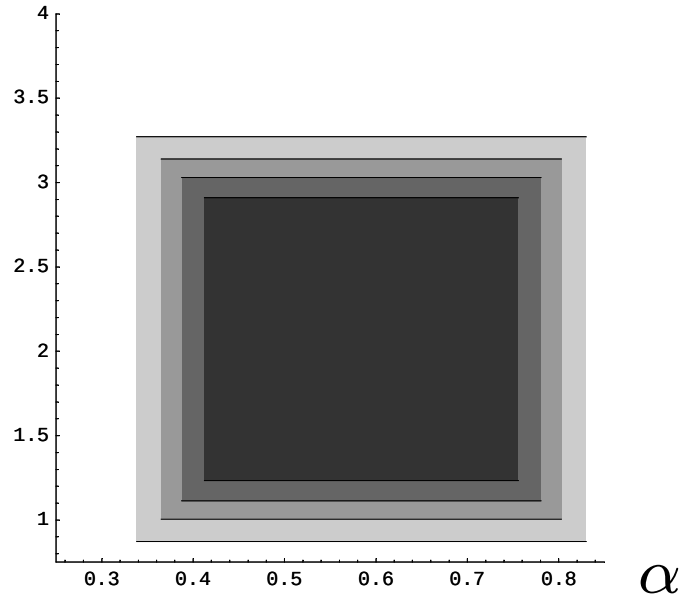
Confidence regions applied to Example 2:

($T = 108$, levels 90 %, 95 %, 97.5 %, 99 %)

(a)



(b)



Method of moments:

$$\widehat{\alpha}_{\mathrm{MM}} = \frac{\sum_{t=2}^{T}(X_t - \bar{X}_T)(X_{t-1} - \bar{X}_T)}{\sum_{t=1}^{T}(X_t - \bar{X}_T)^2},$$

$$\widehat{\beta}_{\mathrm{MM}} = \bar{X}_T \cdot (1 - \widehat{\alpha}_{\mathrm{MM}}).$$

Applying arguments of Theorem 3 in Freeland & McCabe (2005), it follows that MM estimator has same asymptotic properties as CLS estimator.

The INARCH(1) Model

Marginal Distribution

Problem:

As shown above, **marginal cumulants** can be computed recursively up to any order.

But explicit expression for **marginal probabilities** $p_i := P(X_t = i)$ of INARCH(1) process not known!

Feasible approximation of p_i possible?

First approach: Markov chain approximation

$(X_t)_{\mathbb{Z}}$ is ergodic Markov chain, it follows that

$$p_i \;=\; \lim_{n\to\infty} \; p_{i|j}(n) \qquad \text{for all } i,j \in \mathbb{N}_0,$$

where n-step transition probabilities

$$p_{i|j}(n) := P(X_t = i \mid X_{t-n} = j)$$

follow recursively via

$$p_{i|j}(n) \;=\; \sum_{r=0}^{\infty} \; p_{i|r} \cdot p_{r|j}(n-1).$$

INARCH(1) Model: Marginal Distribution

These relations allow to determine marginal probabilities numerically:

Choosing $M, N \in \mathbb{N}$ sufficiently large, we approximate

$$p_i \approx p_{i|j}(N), \qquad \text{where } p_{i|j}(n) \approx \sum_{r=0}^{M} p_{i|r} \cdot p_{r|j}(n-1)$$

for arbitrary $i, j \in \mathbb{N}_0$, e. g., choose $j := \lceil \mu_X \rceil$.

Summary: Precise but computationally expensive, requires appropriate choice of M, N.

Simpler approximation possible?

Second approach:

Poisson-Charlier expansion of Barbour (1987).

Probability generating function (pgf) of X: $p_X(z) := E[z^X]$.

Factorial cumulant generating function (fcgf):

$$k_X(z) \;:=\; \ln\big(p_X(1+z)\big) \;=\; \ln E[(1+z)^X] \;=:\; \sum_{r=1}^{\infty} \frac{\kappa_{(r)}}{r!} \cdot z^r,$$

with factorial cumulants $\kappa_{(r)}$. Relation between pgf and fcgf:

$$p_X(z) \;=\; \exp\big(k_X(z-1)\big) \;=\; \exp\big(\textstyle\sum_{r=1}^{\infty} \frac{\kappa_{(r)}}{r!} \cdot (z-1)^r\big).$$

Basic idea: $p_X(z) \;\approx\; \exp\big(\sum_{r=1}^{m} \frac{\kappa_{(r)}}{r!} \cdot (z-1)^r\big)$.

Poisson-Charlier expansion of Barbour (1987) further refinement of this approach.

Let $\pi_i := e^{-\kappa_1} \cdot \kappa_1^i / i!$ denote Poisson probabilities.
Let ∇ denote difference operator: $\nabla \pi_i = \pi_i - \pi_{i-1}$.

m^{th} order **PC approximation**: $p_i \approx f_m(\nabla) \cdot \pi_i$,
where f_m is $(m-1)^{\text{th}}$ order Taylor polynomial
around $z = 0$ and evaluated in $z = 1$ of

$$f(z, \nabla) \;:=\; \exp\left(\frac{1}{z} \cdot \sum_{r=2}^{\infty} \frac{\kappa_{(r)}}{r!} \cdot (-z\nabla)^r \right).$$

INARCH(1) Model: Marginal Distribution

The first four Poisson-Charlier approximations:

$$
\begin{aligned}
f_1(\nabla) &= 1, \qquad (\rightarrow \text{Poisson approximation}) \\
f_2(\nabla) &= 1 + \tfrac{1}{2}\kappa_{(2)}\nabla^2, \\
f_3(\nabla) &= 1 + \tfrac{1}{2}\kappa_{(2)}\nabla^2 - \tfrac{1}{6}\kappa_{(3)}\nabla^3 + \tfrac{1}{8}\kappa_{(2)}^2\nabla^4, \\
f_4(\nabla) &= 1 + \tfrac{1}{2}\kappa_{(2)}\nabla^2 - \tfrac{1}{6}\kappa_{(3)}\nabla^3 + \left(\tfrac{\kappa_{(2)}^2}{8} + \tfrac{\kappa_{(4)}}{24}\right)\nabla^4 \\
&\quad - \tfrac{1}{12}\kappa_{(2)}\kappa_{(3)}\nabla^5 + \tfrac{1}{48}\kappa_{(2)}^3\nabla^6.
\end{aligned}
$$

So only knowledge about first few factorial cumulants of X required!

Weiß (2009b): (Marginal Factorial Cumulants)

Factorial cumulants of INARCH(1) process determined from usual cumulants via

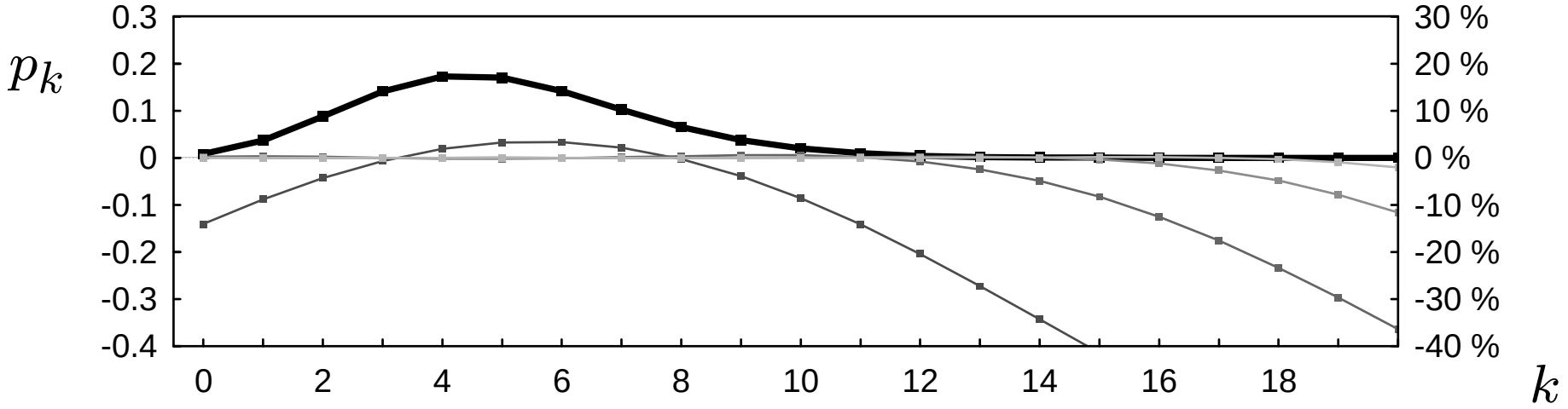
$$\kappa_{(1)} = \kappa_1, \qquad \kappa_{(n)} = \alpha^n \cdot \kappa_n \quad \text{for } n \geq 2.$$

$\Rightarrow$ Poisson-Charlier approximation of any order m easily implemented for INARCH(1) model!

Performance?

INARCH(1) Model: Marginal Distribution

True marginal distribution and relative errors of Poisson-Charlier approximations for $(\beta, \alpha) = (3.75, 0.25)$:



$\Rightarrow$ Approximations only good in central region:

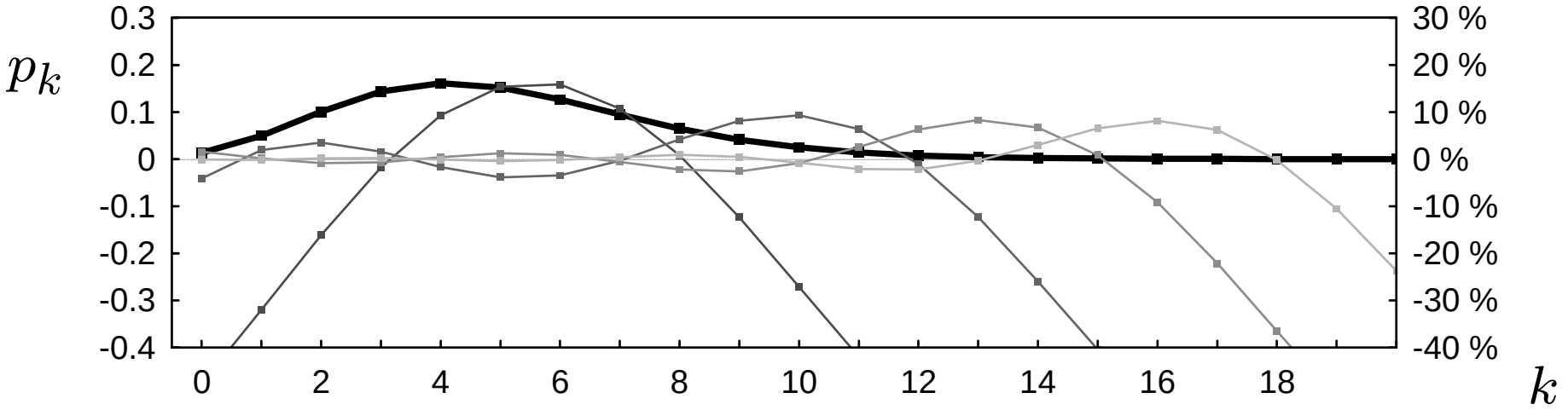
For $k \to \infty$, relative error tends to $-100\,\%$.

Usual Poisson approximation always bad.

INARCH(1) Model: Marginal Distribution

Performance becomes worse for increasing α.

The fourth order approximation still works well for $\alpha = 0.5$ (graph for $\beta = 2.5$):



For $\alpha \geq 0.6$, no satisfactory approximation for the marginal probabilities.

Conclusions and Future Research

- INGARCH models: simple ARMA-like models for time series of overdispersed counts,
  Yule-Walker equations for autocorrelation function.
- INARCH(1) model: Explicit expressions for marginal (factorial) cumulants, autocorrelation function, transition probabilities. Asymptotic distribution for CLS estimators, simultaneous confidence regions.
- Marginal probabilities not explicitly known.
  MC approximation possible but expensive.
  PC approximation better than Poisson approximation, really satisfactorily only for moderate autocorrelation.

Thank You

for Your Interest!

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