# Goodness-of-Fit Testing for Count Time Series


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# Goodness-of-Fit Testing for Count Time Series

## Introduction

In applications, often testing hypothesis w.r.t.
marginal distribution of given count time series $X_1, \ldots, X_T$.

Might be done by

- looking at specific feature of hypothetical count distribution, e. g., Poisson index of dispersion (Schweer & Weiß, 2014);
- deriving test statistic considering any deviation from null model, e. g., based on pgf (Meintanis & Karlis, 2014).

**Textbook approach:**

GoF statistics based on hypothetical & estimated **pmf**,
e. g., power-divergence family (Read & Cressie,1988).

# Goodness-of-Fit Testing for Count T.S.



Many common GoF tests within power-divergence family. Since asymptotic behavior same as of **Pearson's GoF statistic** $G^2$ (Read & Cressie,1988), focus on latter in sequel.

If $X_1, \ldots, X_T$ **i. i. d.**, and if Pearson statistic $G^2$ constructed by using $k$ categories, then asymptotic $\chi^2$-distribution:

- if hypothetical distribution fully specified, then $G^2$ with asymptotic $\chi^2_{k-1}$-distribution;
- if hypothetical distribution has $r \geq 1$ unspecified parameters to be estimated from data, then $G^2$ with asymptotic $\chi^2_{k-1-r}$-distribution.

**Aim:**

- Develop general approach to *explicitly* compute asymptotic distribution of Pearson's GoF statistic for *serially dependent* count time series,
- covers both scenarios, where model parameters *specified or estimated* from available data.

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# Pearson's Goodness-of-Fit Test

Test Statistic

# Pearson's GoF Test: Test Statistic



Let $(X_t)_{\mathbb{Z}}$ stationary count process.

GoF test based on $X_1,\ldots,X_T$ to test **hypothesis** $H_0$**:**

marginal distribution satisfies $P(X_t=i)=p_i$ for some pmf $(p_i)_i$.

Following common rules of thumb (Horn, 1977),

first define appropriate **set of categories**,

e. g., of the form (Kim & Weiß, 2015)

$$\{0,\ldots,a\},\ \{a+1\},\ldots,\ \{b\},\ \{b+1,\ldots\}\quad \text{with some } 0\le a<b.$$

Then $H_0$-probabilities $\boldsymbol{\pi}=(\pi_a,\ldots,\pi_b,\pi_{b+1})^\top$

for $X_t$ falling into these categories

computed from $\boldsymbol{p}=(p_0,\ldots,p_b)^\top$ as (. . . )

# Pearson's GoF Test: Test Statistic

$$\boldsymbol{\pi} := \begin{pmatrix} \pi_a \\ \pi_{a+1} \\ \vdots \\ \pi_b \\ \pi_{b+1} \end{pmatrix} := \begin{pmatrix} p_0 + \ldots + p_a \\ p_{a+1} \\ \vdots \\ p_b \\ 1 - p_0 - \ldots - p_b \end{pmatrix} = \mathbf{A}\,\boldsymbol{p} + \boldsymbol{e}_{b-a+2}$$

with $\boldsymbol{e}_{b-a+2} := (0, \ldots, 0, 1)^\top$, and

$$\mathbf{A} := \mathbf{A}(a,b) := \underbrace{\begin{pmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ -1 & \cdots & -1 \end{pmatrix}}_{a+1}\underbrace{\begin{pmatrix} 0 & \cdots & 0 \\ 1 & & \vdots \\ & \ddots & 0 \\ 0 & & 1 \\ -1 & \cdots & -1 \end{pmatrix}}_{b-a} \Bigg\} b-a+2.$$

# Pearson's GoF Test: Test Statistic



**Estimation** by relative frequencies: $\hat{p}_i = \frac{1}{T}\sum_{t=1}^{T} \mathbb{1}_{\{X_t=i\}}$

and $\hat{\boldsymbol{\pi}} := \mathbf{A}\,\hat{\boldsymbol{p}} + \boldsymbol{e}_{b-a+2}$ with $\hat{\boldsymbol{p}} = (\hat{p}_0, \ldots, \hat{p}_b)^\top$.

**Test statistic** of Pearson's GoF test:

$$\begin{aligned} G^2 &= T\,(\hat{\boldsymbol{\pi}} - \boldsymbol{\pi})^\top \operatorname{diag}(\boldsymbol{\pi})^{-1}\,(\hat{\boldsymbol{\pi}} - \boldsymbol{\pi}) \\ &= T\,\boldsymbol{G}^\top \boldsymbol{G} \qquad \text{with } \boldsymbol{G} := \operatorname{diag}\big(\pi_a^{-1/2}, \ldots, \pi_{b+1}^{-1/2}\big)\,\mathbf{A}(\hat{\boldsymbol{p}} - \boldsymbol{p}). \end{aligned}$$

Typically, marginal pmf determined by some parameter $\boldsymbol{\theta} \in \mathbb{R}^r$,

i. e., $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{\theta})$ and $\boldsymbol{\pi} = \boldsymbol{\pi}(\boldsymbol{\theta}) = \mathbf{A}\,\boldsymbol{p}(\boldsymbol{\theta}) + \boldsymbol{e}_{b-a+2}$.

Pearson's GoF test statistic

$$G^2 \;=\; G^2(\widehat{\boldsymbol{p}}, \boldsymbol{\theta}).$$

**Two scenarios:**

- $\boldsymbol{\theta}$ specified (so fully specified pmf),

  so test statistic $G^2(\widehat{\boldsymbol{p}}, \boldsymbol{\theta})$, or

- $\boldsymbol{\theta}$ estimated from $X_1, \ldots, X_T$ (GoF with estimated par.),

  so test statistic $G^2(\widehat{\boldsymbol{p}}, \widehat{\boldsymbol{\theta}})$.

For estimation, we use simple moment estimators, i. e.,

$$\boldsymbol{\theta} = \boldsymbol{h}(\mu_1, \ldots, \mu_r) \qquad \text{and} \qquad \widehat{\boldsymbol{\theta}} = \boldsymbol{h}(\widehat{\mu}_1, \ldots, \widehat{\mu}_r),$$

where $\mu_k := E[X_t^k]$ with $\mu := \mu_1$

and $\widehat{\mu}_k = \frac{1}{T}\sum_{t=1}^{T} X_t^k$ with $\widehat{\mu}_1 = \bar{X}$.

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# Pearson's Goodness-of-Fit Test

Asymptotics

# Pearson's GoF Test: Asymptotics



**Aim:** Derive asymptotic distributions of $\sqrt{T}\,\boldsymbol{G}(\hat{\boldsymbol{p}},\boldsymbol{\theta})$ and $\sqrt{T}\,\boldsymbol{G}(\hat{\boldsymbol{p}},\hat{\boldsymbol{\theta}})$, respectively, as well as of $G^2(\hat{\boldsymbol{p}},\boldsymbol{\theta})$ and $G^2(\hat{\boldsymbol{p}},\hat{\boldsymbol{\theta}})$.

**Step 1:** define $(b+1+r)$-dimensional process

$$\boldsymbol{Z}_t = \begin{pmatrix} \mathbb{1}_{\{X_t=0\}} \\ \vdots \\ \mathbb{1}_{\{X_t=b\}} \\ X_t \\ \vdots \\ X_t^r \end{pmatrix} \quad \text{with} \quad \boldsymbol{\mu_Z} := E[\boldsymbol{Z}_t] = \begin{pmatrix} p_0 \\ \vdots \\ p_b \\ \mu_1 \\ \vdots \\ \mu_r \end{pmatrix},$$

**Assumptions:**

- $(X_t)_{\mathbb{Z}}$ is $\alpha$-mixing with geometrically decreasing weights, and $(2r+\delta)$-moments with some $\delta > 0$ exist.
- [Property "SYM": $P(X_t = i, X_{t-h} = j)$ symmetric in $i,j$.]

# Pearson's GoF Test: Asymptotics



Under above conditions, with Theorem 1.7 in Ibragimov (1962):

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\boldsymbol{Z}_t-\boldsymbol{\mu}_{\boldsymbol{Z}}) \xrightarrow{D} \mathsf{N}(\boldsymbol{0},\boldsymbol{\Sigma}),$$

where

$$\begin{aligned}\sigma_{ij} &= E[Z_{0,i}\cdot Z_{0,j}]-\mu_{\boldsymbol{Z},i}\,\mu_{\boldsymbol{Z},j} \\ &\quad + \sum_{h=1}^{\infty}\big(E[Z_{0,i}\cdot Z_{h,j}] + E[Z_{h,i}\cdot Z_{0,j}]-2\,\mu_{\boldsymbol{Z},i}\,\mu_{\boldsymbol{Z},j}\big) \\ &\overset{\text{SYM}}{=} E[Z_{0,i}\cdot Z_{0,j}]-\mu_{\boldsymbol{Z},i}\,\mu_{\boldsymbol{Z},j} + 2\sum_{h=1}^{\infty}\big(E[Z_{h,i}\cdot Z_{0,j}]-\mu_{\boldsymbol{Z},i}\,\mu_{\boldsymbol{Z},j}\big).\end{aligned}$$

Under above conditions, with Theorem 1.7 in Ibragimov (1962):

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} (\boldsymbol{Z}_t - \boldsymbol{\mu}_{\boldsymbol{Z}}) \ \xrightarrow{D}\ \mathsf{N}(\mathbf{0}, \boldsymbol{\Sigma}).$$

If $\boldsymbol{\theta}$ specified:

$$\sqrt{T}\,(\hat{\boldsymbol{p}} - \boldsymbol{p}) \ \xrightarrow{D}\ \mathsf{N}\big(\mathbf{0}, \boldsymbol{\Sigma}^{(\boldsymbol{p})}\big) \qquad \text{with } \boldsymbol{\Sigma}^{(\boldsymbol{p})} := (\sigma_{ij})_{i,j=0,\ldots,b}.$$

If $\boldsymbol{\theta}$ estimated, $\hat{\boldsymbol{\theta}} = \boldsymbol{h}(\hat{\mu}_1, \ldots, \hat{\mu}_r)$:

$$\sqrt{T}\,\big((\hat{\boldsymbol{p}}, \hat{\boldsymbol{\theta}}) - (\boldsymbol{p}, \boldsymbol{\theta})\big) \ \xrightarrow{D}\ \mathsf{N}\big(\mathbf{0}, \boldsymbol{\Sigma}^*\big) \qquad \text{with } \boldsymbol{\Sigma}^* := \mathbf{D}\boldsymbol{\Sigma}\mathbf{D}^\top,$$

with $\mathbf{D}$ denoting the Jacobian of $\big(z_0, \ldots, z_b, \boldsymbol{h}(z_{b+1}, \ldots, z_{b+r})\big)^\top$ evaluated at $(p_0, \ldots, \mu_r)^\top$.

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# Application and Implementation

Intermezzo

**Crucial question for practice:** Possible to compute covariances $\sigma_{ij}$ of asymptotic $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\boldsymbol{Z}_t - \boldsymbol{\mu}_{\boldsymbol{Z}}) \xrightarrow{D} \mathrm{N}(\boldsymbol{0}, \boldsymbol{\Sigma})$?

**Yes,** if $h$-step-ahead conditional probabilities $p_{i|j}{}^{(h)} := P(X_t = i \mid X_{t-h} = j)$ available; property SYM further simplifies formulae.

For $i, j \in \{0, \ldots, b\}$, we have

$$E[Z_{h,i} \cdot Z_{0,j}] - \mu_{\boldsymbol{Z},i}\mu_{\boldsymbol{Z},j} = \begin{cases} (\delta_{i,j} - p_i)\, p_j & \text{if } h = 0, \\ (p_{i|j}^{(h)} - p_i)\, p_j & \text{if } h > 0, \end{cases}$$

# Pearson's GoF Test: Implementation



For $i, j \in \{1, \ldots, r\}$, we have $\mu_{\boldsymbol{Z},b+i} = \mu_i$, $\mu_{\boldsymbol{Z},b+j} = \mu_j$, and

$$E[Z_{h,b+i} \cdot Z_{0,b+j}] \;=\; E[X_t^i \cdot X_{t-h}^j] \;=\; \begin{cases} \mu_{i+j} & \text{if } h = 0, \\ E[X_t^i \cdot X_{t-h}^j] & \text{if } h > 0. \end{cases}$$

Joint moments $E[X_t^i \cdot X_{t-h}^j]$ often available in practice.

If $r = 1$ (one-parameter $H_0$-distribution),

sufficient to compute the autocovariance function.

If no closed-form moment expressions are not available,

then numerical computation from ${p_{i|j}}^{(h)}$, e. g., by truncating

summation after $M + 1$ summands with $M$ sufficiently large.

# Pearson's GoF Test: Implementation

For $i = 0, \ldots, b$ and $j = 1, \ldots, r$,

$$E[Z_{h,b+j} \cdot Z_{0,i}] = \begin{cases} i^j\, p_i & \text{if } h = 0, \\ E[X_t^j \mid X_{t-h} = i]\, p_i & \text{if } h > 0. \end{cases}$$

If SYM hold, then

$$E[Z_{h,i} \cdot Z_{0,b+j}] \overset{\text{SYM}}{=} E[Z_{h,b+j} \cdot Z_{0,i}].$$

If SYM does not hold, then $E[Z_{h,i} \cdot Z_{0,b+j}] = \sum_{x=0}^{\infty} x^j\, p_{i|x}^{(h)}\, p_x$ calculated (numerically) from ${p_{i|j}}^{(h)}$.

**Couple of relevant examples:** $(\ldots)$

# Pearson's GoF Test: Implementation

- **i. i. d. counts:** ✔
- **CLAR(1) model** (Grunwald et al., 2000):

  We have $E[Z_{h,b+1} \cdot Z_{0,i}] = \big(\alpha^h \cdot i + (1-\alpha^h)\,\mu\big)\,p_i$.

  If also SYM holds, then $\sigma_{i,b+1} = (i-\mu)\,p_i\,\frac{1+\alpha}{1-\alpha}$.

  We also have $\rho(h) = \alpha^h$, so

$$\sigma_{b+1,b+1} = \sigma^2 + 2\sum_{h=1}^{\infty} \sigma^2\,\alpha^h = \sigma^2\,\frac{1+\alpha}{1-\alpha}.$$

- **Poisson INAR(1) model** (McKenzie, 1985):

  One-parameter ($r = 1$) marginal $p_i = e^{-\mu}\,\mu^i/i!$, $\hat{\mu} = \bar{X}$;

  CLAR(1) model with property SYM.

# Pearson's GoF Test: Implementation



- **Poisson INAR(1) model** (McKenzie, 1985):

  One-parameter ($r = 1$) marginal $p_i = e^{-\mu}\,\mu^i/i!$, $\hat{\mu} = \bar{X}$;

  CLAR(1) model with property SYM.

  For $i, j \in \{0, \ldots, b\}$,

$$\begin{aligned}\sigma_{ij} &= (\delta_{i,j} - p_i)\,p_j \;+\; 2\,p_j \textstyle\sum_{h=1}^{\infty}(p_{i|j}{}^{(h)} - p_i) \\ &\approx (\delta_{i,j} - p_i)\,p_j \;+\; 2\,p_j \textstyle\sum_{h=1}^{M}(p_{i|j}{}^{(h)} - p_i),\end{aligned}$$

  where simple closed-form formula for $p_{i|j}{}^{(h)}$.

  Computationally more efficient to utilize that

  $(X_t, X_{t-h}) \;\sim\; \text{BPoi}\big(\alpha^h\,\mu;\; (1-\alpha^h)\,\mu, (1-\alpha^h)\,\mu\big)$.

- **Binomial AR(1) model** (McKenzie, 1985):
  CLAR(1) model with property SYM (like before),
  one-parameter $(r = 1)$ marginal $p_i = \binom{n}{i}\,\pi^i\,(1-\pi)^{n-i}$,
  moment estimator $\hat{\pi} := \bar{X}/n$, so
  Jacobian $\mathbf{D} = \mathrm{diag}(1,\ldots,1,\ 1/n)$.
- **Geometric INAR(1) model** (McKenzie, 1985):
  CLAR(1) model, but property SYM does not hold!
  Numerical approximation for ${p_{i|j}}^{(h)}$ and $\sigma_{i,b+1}$.
  One-parameter $(r = 1)$ marginal $p_i = \pi\,(1-\pi)^i$,
  moment estimator $\hat{\pi} = 1/(1+\bar{X})$, so
  Jacobian $\mathbf{D} = \mathrm{diag}\big(1,\ldots,1,\ -\pi^2\big)$.

# Pearson's GoF Test: Implementation

- **Poisson INAR(2) model** (Alzaid & Al-Osh, 1990):

  Poisson marginal, property SYM;

  all computations like for Poisson INAR(1), but

  $(X_t, X_{t-h}) \sim \text{BPoi}\Big(\rho(h)\,\mu;\ \big(1-\rho(h)\big)\,\mu,\ \big(1-\rho(h)\big)\,\mu\Big)$,

  $E[X_t \mid X_{t-h}] = \rho(h)\,X_{t-h} + \big(1-\rho(h)\big)\,\mu$,

  $\rho(1) = \alpha_1$ and $\rho(h) = \alpha_1\,\rho(h-1) + \alpha_2\,\rho(h-2)$ for $h \geq 2$.

- **Poisson INMA(q) model** (Weiß, 2008):

  non-Markovian but $q$-dependent.

  $(X_t, X_{t-h}) \sim \text{BPoi}\Big(\rho(h)\,\mu;\ \big(1-\rho(h)\big)\,\mu,\ \big(1-\rho(h)\big)\,\mu\Big)$,

  requires $\mu$ and $\rho(1), \ldots, \rho(q)$.

- **NDARMA(p, q) model** (Jacobs & Lewis, 1983):

  Defining $c := 1 + 2\sum_{h=1}^{\infty} \rho(h)$,

  $\sigma_{ij} = c\,(\delta_{i,j} - p_i)\,p_j$,

  $\sigma_{i,b+j} = c\,(i^j - \mu_j)\,p_i$,

  $\sigma_{b+i,b+j} = c\,(\mu_{i+j} - \mu_i\,\mu_j)$.

  Corresponds to i. i. d.-case with additional factor $c$.

- **Hidden-Markov model** (Zucchini & MacDonald, 2009):

  Marginal and lagged bivariate probabilities as

$$P(X_t = i) \;=\; \mathbf{1}^\top \mathbf{P}(i)\,\boldsymbol{\pi},$$

$$P(X_t = i, X_{t-h} = j) \;=\; \mathbf{1}^\top \mathbf{P}(i)\,\mathbf{A}^h\,\mathbf{P}(j)\,\boldsymbol{\pi}.$$

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# Pearson's Goodness-of-Fit Test

Asymptotics

# Pearson's GoF Test: Asymptotics



Let's return to asymptotics $\sqrt{T}\big((\hat{\boldsymbol{p}},\hat{\boldsymbol{\theta}})-(\boldsymbol{p},\boldsymbol{\theta})\big)$.

**Step 2:** derive asymptotics of $\sqrt{T}\,\boldsymbol{G}(\hat{\boldsymbol{p}},\boldsymbol{\theta})$ and $\sqrt{T}\,\boldsymbol{G}(\hat{\boldsymbol{p}},\hat{\boldsymbol{\theta}})$.

Let $\boldsymbol{z}=(\boldsymbol{z}_1,\boldsymbol{z}_2)$ with $\boldsymbol{z}_1\in\mathbb{R}^{b+1}$ and $\boldsymbol{z}_2\in\mathbb{R}^r$, define

$$\boldsymbol{g}(\boldsymbol{z}_1,\boldsymbol{z}_2) \;:=\; \text{diag}\big(\pi_a(\boldsymbol{z}_2)^{-1/2},\ldots,\pi_{b+1}(\boldsymbol{z}_2)^{-1/2}\big)\,\mathbf{A}\big(\boldsymbol{z}_1-\boldsymbol{p}(\boldsymbol{z}_2)\big)$$

such that $\boldsymbol{G}(\hat{\boldsymbol{p}},\boldsymbol{\theta})=\boldsymbol{g}(\hat{\boldsymbol{p}},\boldsymbol{\theta})$ and $\boldsymbol{G}(\hat{\boldsymbol{p}},\hat{\boldsymbol{\theta}})=\boldsymbol{g}(\hat{\boldsymbol{p}},\hat{\boldsymbol{\theta}})$.

Delta method requires Jacobian of $\boldsymbol{g}$.

- In specified-parameter case, reduced Jacobian suffices,

$$\mathbf{J}_g(\boldsymbol{z}_1) \;=\; \big(\tfrac{\partial}{\partial z_{1,i}}\, g_k(\boldsymbol{z}_1, \boldsymbol{\theta})\big)_{k=a,\dots,b+1,\; i=0,\dots,b},$$

leading to

$$\sqrt{T}\, \boldsymbol{G}(\widehat{\boldsymbol{p}}, \boldsymbol{\theta}) \;\xrightarrow{D}\; \mathsf{N}\Big(\mathbf{0}, \mathbf{D}_{\mathsf{kn}} \boldsymbol{\Sigma}^{(\boldsymbol{p})} \mathbf{D}_{\mathsf{kn}}^\top\Big)$$

with $\mathbf{D}_{\mathsf{kn}} \;=\; \mathrm{diag}\Big(\boldsymbol{\pi}(\boldsymbol{\theta})^{-1/2}\Big)\, \mathbf{A}$.

# Pearson's GoF Test: Asymptotics

- In estimated-parameter case, full Jacobian required.

  Denoting Jacobian of $\boldsymbol{p}(\boldsymbol{z}_2)$ by $\mathbf{J}_{\boldsymbol{p}}(\boldsymbol{z}_2)$, we obtain

$$\sqrt{T}\,\boldsymbol{G}(\widehat{\boldsymbol{p}},\widehat{\boldsymbol{\theta}}) \;\xrightarrow{D}\; \mathsf{N}\Big(\mathbf{0},\mathbf{D}_{\mathsf{est}}\boldsymbol{\Sigma}^*\mathbf{D}_{\mathsf{est}}^\top\Big)$$

  with $\mathbf{D}_{\mathsf{est}} \;=\; \Big(\mathbf{D}_{\mathsf{kn}},\; -\mathbf{D}_{\mathsf{kn}}\,\mathbf{J}_{\boldsymbol{p}}(\boldsymbol{\theta})\Big)$.

**Remark:** Jacobian $\mathbf{J}_{\boldsymbol{p}}$ easily computed for

- Poisson marginal via $\frac{\partial}{\partial\mu}\,p_i(\mu) \;=\; p_{i-1}(\mu) - p_i(\mu)$,
- binomial marg. via $\frac{\partial}{\partial\pi}\,p_{n,i}(\pi) \;=\; n\,\big(p_{n-1,i-1}(\pi) - p_{n-1,i}(\pi)\big)$,
- geometric marginal via $\frac{\partial}{\partial\pi}\,p_i(\pi) \;=\; \frac{1}{\pi}\,p_i(\pi) - i\,p_{i-1}(\pi)$.

**Step 3:** We apply Theorem 3.1 in Tan (1977), leading to asymptotic quadratic-form distribution for Pearson's GoF test statistic.

- Specified-parameter case:

$$G^2(\hat{\boldsymbol{p}}, \boldsymbol{\theta}) \xrightarrow{D} \sum_{i=1}^{u} \lambda_i\, Z_i^2,$$

  where $\lambda_1, \ldots, \lambda_u$ non-zero eigenvalues of $\mathbf{D}_{\mathsf{kn}} \boldsymbol{\Sigma}^{(\boldsymbol{p})} \mathbf{D}_{\mathsf{kn}}^\top$.

- Estimated-parameter case:

$$G^2(\hat{\boldsymbol{p}}, \hat{\boldsymbol{\theta}}) \xrightarrow{D} \sum_{j=1}^{v} \lambda_j^*\, Z_j^2,$$

  where $\lambda_1^*, \ldots, \lambda_v^*$ non-zero eigenvalues of $\mathbf{D}_{\mathsf{est}} \boldsymbol{\Sigma}^* \mathbf{D}_{\mathsf{est}}^\top$.

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# Pearson's Goodness-of-Fit Test

Computations & Simulations

Implementation of asymptotics in R,

using `CompQuadForm` package (Duchesne & de Micheaux, 2010)

to evaluate quadratic-form distributions.

Simulations always use 10 000 replications.

Quantiles of Pearson statistic (estimated parameters)

for binomial AR(1) process, asymptotic vs. simulated values,

where latter (italic) taken from Kim & Weiß (2015):

| $n$ | $\pi$ | $\rho$ | $T$ | $a$ | $b$ | $q_{0.25}$ | | $q_{0.50}$ | | $q_{0.75}$ | | $q_{0.95}$ | | $q_{0.99}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0.4 | 0.00 | 70 | 0 | 4 | 1.923 | *1.914* | 3.357 | *3.268* | 5.385 | *5.246* | 9.49 | *9.28* | 13.28 | *13.66* |
| 5 | 0.4 | 0.50 | 70 | 0 | 4 | 2.449 | *2.351* | 4.292 | *3.962* | 6.921 | *6.293* | 12.33 | *11.73* | 17.44 | *18.19* |
| 22 | 0.4 | 0.00 | 70 | 5 | 11 | 3.483 | *3.554* | 5.377 | *5.418* | 7.870 | *7.903* | 12.62 | *12.59* | 16.84 | *16.65* |
| 22 | 0.4 | 0.50 | 70 | 5 | 11 | 3.845 | *3.911* | 5.951 | *6.000* | 8.744 | *8.621* | 14.14 | *13.78* | 19.03 | *18.80* |

Properties of Pearson statistic (estimated parameters)

for Poisson INAR(1) process with $\mu = 3$,

asymptotic vs. simulated values:

| $\alpha$ | $T$ | $a$ | $b$ | mean | | std. dev. | | $q_{0.25}$ | | $q_{0.50}$ | | $q_{0.75}$ | | $q_{0.95}$ | | $q_{0.99}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.25 | 200 | 0 | 6 | 6.15 | 6.13 | 3.55 | 3.50 | 3.54 | 3.59 | 5.48 | 5.45 | 8.03 | 7.94 | 12.90 | 12.78 | 17.23 | 17.34 |
| | 500 | 0 | 7 | 7.16 | 7.14 | 3.83 | 3.81 | 4.35 | 4.33 | 6.49 | 6.46 | 9.24 | 9.22 | 14.39 | 14.41 | 18.91 | 18.73 |
| | 1000 | 0 | 8 | 8.17 | 8.18 | 4.09 | 4.08 | 5.17 | 5.23 | 7.49 | 7.53 | 10.43 | 10.36 | 15.83 | 15.73 | 20.52 | 20.75 |
| 0.50 | 200 | 0 | 6 | 6.89 | 6.86 | 4.04 | 3.96 | 3.93 | 4.00 | 6.11 | 6.10 | 9.00 | 8.93 | 14.59 | 14.20 | 19.68 | 19.37 |
| | 500 | 0 | 7 | 7.98 | 7.94 | 4.34 | 4.33 | 4.81 | 4.81 | 7.20 | 7.18 | 10.30 | 10.26 | 16.19 | 15.91 | 21.48 | 21.17 |
| | 1000 | 0 | 8 | 9.05 | 8.95 | 4.61 | 4.64 | 5.69 | 5.59 | 8.27 | 8.17 | 11.56 | 11.36 | 17.71 | 17.64 | 23.16 | 23.45 |
| 0.75 | 200 | 0 | 6 | 10.49 | 10.16 | 6.68 | 6.35 | 5.72 | 5.74 | 9.04 | 8.84 | 13.65 | 13.12 | 23.29 | 21.88 | 32.82 | 31.66 |
| | 500 | 0 | 7 | 11.99 | 11.69 | 7.12 | 7.02 | 6.90 | 6.84 | 10.52 | 10.25 | 15.44 | 14.81 | 25.55 | 24.64 | 35.42 | 35.68 |
| | 1000 | 0 | 8 | 13.40 | 13.32 | 7.46 | 7.62 | 8.06 | 8.01 | 11.93 | 11.80 | 17.11 | 16.74 | 27.55 | 27.44 | 37.64 | 39.53 |

Properties of Pearson statistic (estimated parameters)

for geometric INAR(1) process with $\mu = 3$,

asymptotic vs. simulated values:

| $\alpha$ | $T$ | $a$ | $b$ | mean | | std. dev. | | $q_{0.25}$ | | $q_{0.50}$ | | $q_{0.75}$ | | $q_{0.95}$ | | $q_{0.99}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.25 | 200 | 0 | 8 | 8.25 | 8.32 | 4.11 | 4.14 | 5.24 | 5.30 | 7.57 | 7.63 | 10.52 | 10.58 | 15.97 | 16.05 | 20.73 | 20.67 |
| | 500 | 0 | 11 | 11.21 | 11.19 | 4.79 | 4.78 | 7.72 | 7.71 | 10.53 | 10.53 | 13.95 | 13.95 | 20.07 | 19.90 | 25.26 | 25.33 |
| | 1000 | 0 | 13 | 13.19 | 13.28 | 5.19 | 5.31 | 9.43 | 9.42 | 12.51 | 12.67 | 16.22 | 16.41 | 22.73 | 22.92 | 28.19 | 28.59 |
| 0.50 | 200 | 0 | 8 | 9.29 | 9.26 | 4.88 | 4.79 | 5.77 | 5.77 | 8.41 | 8.39 | 11.83 | 11.81 | 18.48 | 18.09 | 24.71 | 24.46 |
| | 500 | 0 | 11 | 12.25 | 12.35 | 5.48 | 5.54 | 8.29 | 8.38 | 11.39 | 11.45 | 15.25 | 15.32 | 22.44 | 22.84 | 28.94 | 29.48 |
| | 1000 | 0 | 13 | 14.23 | 14.34 | 5.85 | 5.85 | 10.02 | 10.17 | 13.39 | 13.51 | 17.51 | 17.57 | 25.02 | 25.21 | 31.69 | 31.70 |
| 0.75 | 200 | 0 | 8 | 14.86 | 14.67 | 8.97 | 8.70 | 8.59 | 8.52 | 12.87 | 12.87 | 18.87 | 18.72 | 31.96 | 31.51 | 45.58 | 43.51 |
| | 500 | 0 | 11 | 18.03 | 18.09 | 9.49 | 9.60 | 11.39 | 11.42 | 16.10 | 16.04 | 22.47 | 22.53 | 35.99 | 36.35 | 49.89 | 50.67 |
| | 1000 | 0 | 13 | 20.09 | 20.22 | 9.78 | 9.91 | 13.24 | 13.27 | 18.20 | 18.20 | 24.77 | 24.99 | 38.50 | 39.14 | 52.50 | 53.28 |

Properties of Pearson statistic (estimated parameters) for Poisson INAR(2) process with $\mu = 3$, asymptotic vs. simulated values:

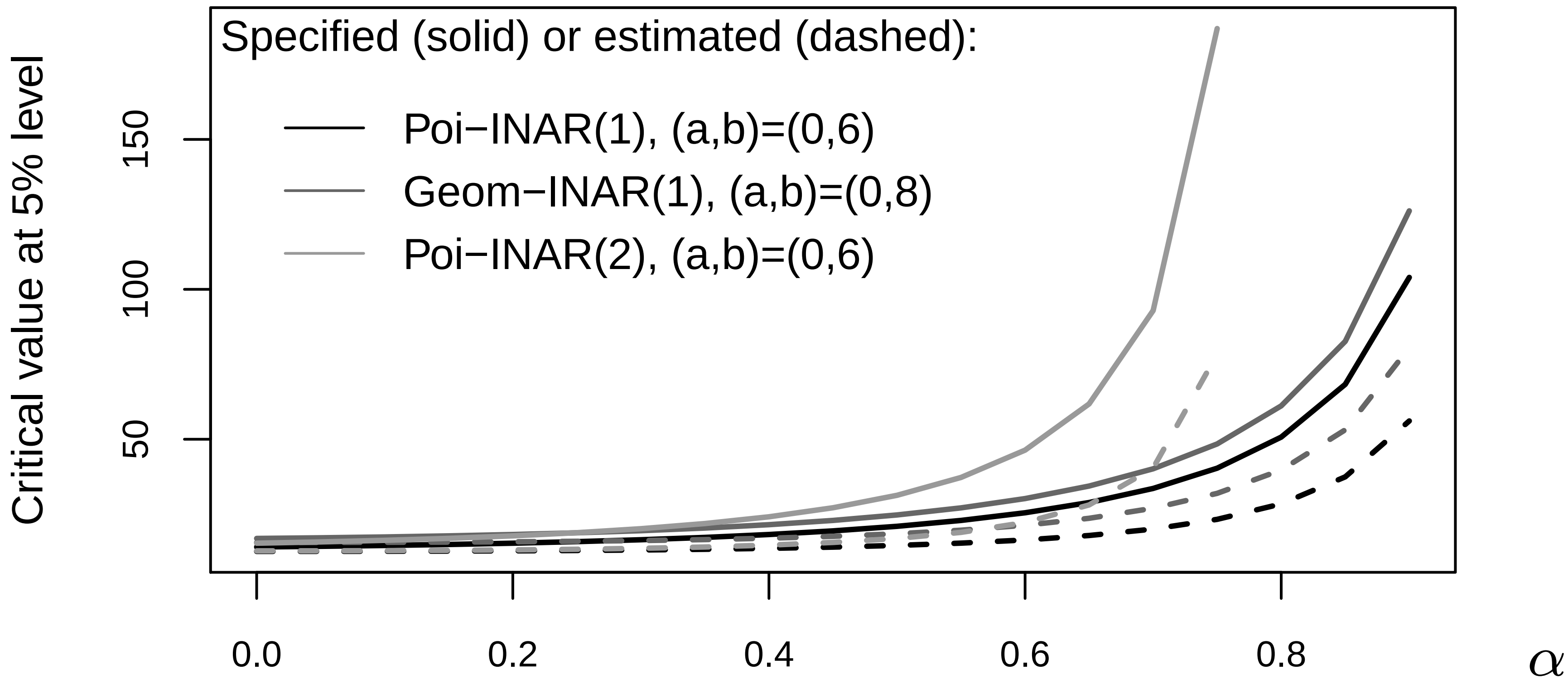
| $\alpha_1$ | $\alpha_2$ | $T$ | $a$ | $b$ | mean | | std. dev. | | $q_{0.25}$ | | $q_{0.50}$ | | $q_{0.75}$ | | $q_{0.95}$ | | $q_{0.99}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.25 | 0.25 | 200 | 0 | 6 | 6.44 | 6.35 | 3.74 | 3.71 | 3.70 | 3.64 | 5.73 | 5.68 | 8.41 | 8.28 | 13.56 | 13.32 | 18.18 | 18.17 |
| | | 500 | 0 | 7 | 7.48 | 7.43 | 4.03 | 4.04 | 4.53 | 4.47 | 6.76 | 6.71 | 9.65 | 9.58 | 15.09 | 15.06 | 19.89 | 19.79 |
| | | 1000 | 0 | 8 | 8.50 | 8.49 | 4.28 | 4.34 | 5.37 | 5.33 | 7.79 | 7.68 | 10.86 | 10.84 | 16.55 | 16.42 | 21.52 | 22.12 |
| 0.50 | 0.25 | 200 | 0 | 6 | 8.36 | 8.11 | 5.25 | 5.00 | 4.60 | 4.58 | 7.24 | 7.10 | 10.86 | 10.46 | 18.37 | 17.61 | 25.80 | 24.98 |
| | | 500 | 0 | 7 | 9.61 | 9.37 | 5.61 | 5.65 | 5.59 | 5.49 | 8.48 | 8.15 | 12.36 | 11.84 | 20.24 | 19.60 | 27.96 | 29.49 |
| | | 1000 | 0 | 8 | 10.79 | 10.68 | 5.89 | 5.93 | 6.57 | 6.53 | 9.66 | 9.59 | 13.75 | 13.41 | 21.91 | 21.77 | 29.79 | 30.66 |
| 0.75 | 0.10 | 200 | 0 | 6 | 13.60 | 12.84 | 9.37 | 8.91 | 7.04 | 6.97 | 11.35 | 10.84 | 17.63 | 16.29 | 31.64 | 28.74 | 46.20 | 45.19 |
| | | 500 | 0 | 7 | 15.44 | 14.86 | 9.96 | 10.01 | 8.45 | 8.24 | 13.15 | 12.62 | 19.85 | 18.69 | 34.56 | 32.74 | 49.67 | 50.32 |
| | | 1000 | 0 | 8 | 17.12 | 16.65 | 10.39 | 10.70 | 9.80 | 9.47 | 14.82 | 14.29 | 21.85 | 21.00 | 37.01 | 36.05 | 52.43 | 53.78 |

# Pearson's GoF Test: Computations & Simulations



Critical values w.r.t. Poisson or geometric marginal distribution having mean $\mu = 3$, sample size $T = 200$;

INAR(2) process satisfies $\alpha_1 = \alpha, \alpha_2 = 0.2$:



Simulated size of Pearson test (level 5 %) for Poisson INAR(1), Poisson INAR(2) (also if misspecified as Poisson INAR(1)), and geometric INAR(1) process, all having mean $\mu = 3$:

| $\alpha =$ $\alpha_1$ | $\alpha_2$ | $T$ | $a$ | $b$ | Poi-INAR(1) | Poi-INAR(2) | misspec. as Poi-INAR(1) | $a$ | $b$ | Geom-INAR(1) |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.25 | 0.25 | 200 | 0 | 6 | 0.050 | 0.046 | 0.058 | 0 | 8 | 0.049 |
| | | 500 | 0 | 7 | 0.046 | 0.049 | 0.062 | 0 | 11 | 0.052 |
| | | 1000 | 0 | 8 | 0.051 | 0.047 | 0.059 | 0 | 13 | 0.051 |
| 0.50 | 0.25 | 200 | 0 | 6 | 0.049 | 0.055 | 0.104 | 0 | 8 | 0.055 |
| | | 500 | 0 | 7 | 0.045 | 0.044 | 0.100 | 0 | 11 | 0.051 |
| | | 1000 | 0 | 8 | 0.048 | 0.048 | 0.110 | 0 | 13 | 0.051 |
| 0.75 | 0.10 | 200 | 0 | 6 | 0.049 | 0.076 | 0.132 | 0 | 8 | 0.078 |
| | | 500 | 0 | 7 | 0.047 | 0.057 | 0.125 | 0 | 11 | 0.058 |
| | | 1000 | 0 | 8 | 0.047 | 0.052 | 0.127 | 0 | 13 | 0.052 |

# Pearson's GoF Test: Computations & Simulations



Simulated size and power of Pearson test (level 5 %) for Poisson INAR(1) process (dispersion index $I = 1$) and NB-INAR(1) process ($I > 1$), respectively, all having mean $\mu = 3$:

| | | | | Rejection rates; $I =$ | | | | |
|---|---|---|---|---|---|---|---|---|
| $\alpha$ | $T$ | $a$ | $b$ | 1 | 1.05 | 1.10 | 1.20 | 1.50 |
| 0.25 | 200 | 0 | 6 | 0.050 | 0.065 | 0.094 | 0.228 | 0.796 |
| | 500 | 0 | 7 | 0.046 | 0.085 | 0.174 | 0.537 | 0.997 |
| | 1000 | 0 | 8 | 0.051 | 0.120 | 0.325 | 0.857 | 1.000 |
| 0.50 | 200 | 0 | 6 | 0.049 | 0.060 | 0.082 | 0.179 | 0.695 |
| | 500 | 0 | 7 | 0.045 | 0.082 | 0.160 | 0.460 | 0.987 |
| | 1000 | 0 | 8 | 0.048 | 0.118 | 0.288 | 0.769 | 1.000 |
| 0.75 | 200 | 0 | 6 | 0.049 | 0.052 | 0.064 | 0.121 | 0.458 |
| | 500 | 0 | 7 | 0.047 | 0.065 | 0.104 | 0.286 | 0.878 |
| | 1000 | 0 | 8 | 0.047 | 0.090 | 0.189 | 0.529 | 0.995 |

# Conclusions



- Distribution of Pearson's GoF test statistic, if applied to count time series, asymptotically approximated by quadratic-form distribution.
- Distribution can be explicitly computed for variety of practically relevant count process models.
- Covers both null model being fully specified, and where parameters have to be estimated.
- Simulations: asymptotic approximation works rather well, test successfully applied to uncover model violations.

# Thank You

# for Your Interest!

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