Binomial Autoregressive Processes with Density-Dependent Thinning

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MATH STAT

Time Series of Counts with a Finite Range

Motivation & Outline

Time Series of Counts with a Finite Range

During last 30 years,

increasing research interest in count data time series.

Initially, research focussed nearly exclusively

on counts with unlimited range $\mathbb{N}_0$.

Only during last years, considerable interest also

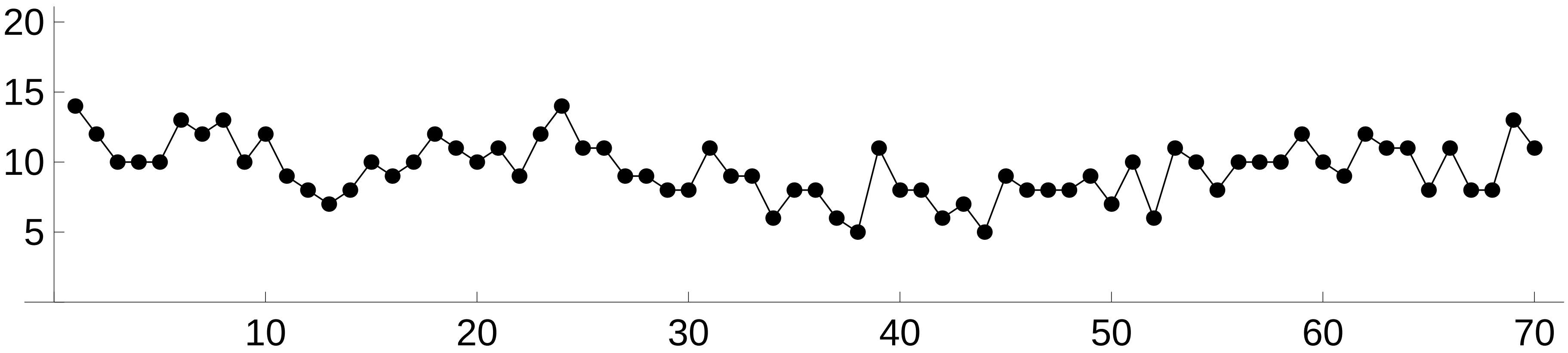
in time series with **fixed finite range** $\{0, 1, \ldots, N\}$.

Examples:

- utilization of computer pools (with N workstations);
- spread of metapopulations (with N patches);
- . . .

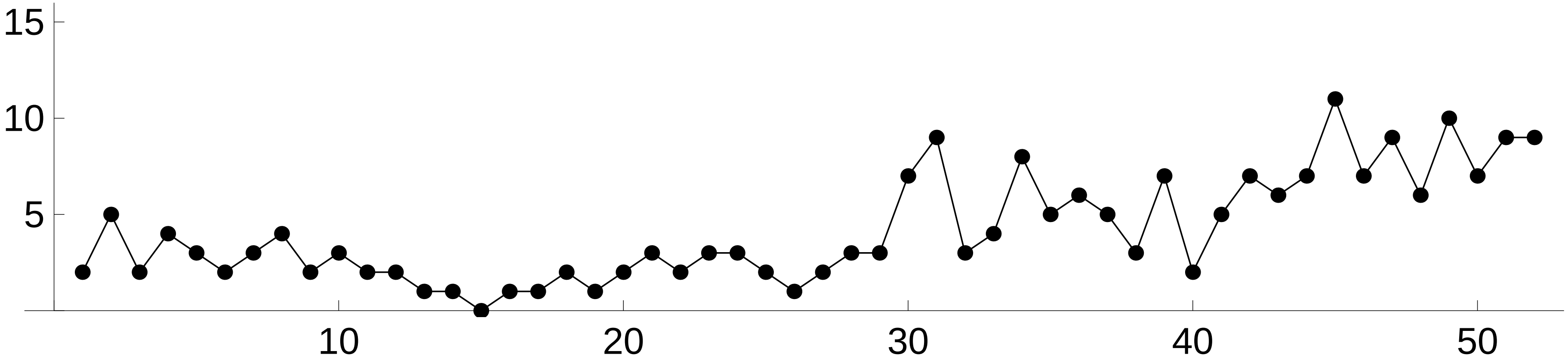
Data example 1: (Weiß & Kim, 2013)

Number of securities companies (among $N = 22$ companies) traded in Korea stock market per 5-min period (Feb. 8, 2011, 09:00–14:50; $T = 70$).



Data example 2: (Robert-Koch-Institut, `survstat.rki.de`)

Number of districts in Germany (among $N = 38$ districts) with new case of hantavirus infection (weekly data, 2011; $T = 52$).



Modeling of time series with fixed finite range $\{0, 1, \ldots, N\}$?

Popular **basic approach** (counterpart to AR(1) model):
binomial AR(1) model (BAR(1)) by McKenzie (1985),
which uses **binomial thinning** operation
$\alpha \circ X \ \sim\ \text{Bin}(X, \alpha)$ by Steutel & van Harn (1979).

Parameters $\pi \in (0; 1)$, $\rho \in \left(\max\{-\frac{\pi}{1-\pi}, -\frac{1-\pi}{\pi}\}\ ;\ 1\right)$,
define thinning probabilities $\beta := \pi\,(1-\rho)$ and $\alpha := \beta + \rho$.
BAR(1) recursion

$$X_{t+1} \;=\; \underbrace{\alpha \circ X_t}_{\text{survivors}} + \underbrace{\beta \circ (N - X_t)}_{\text{newly occupied}},$$

thinnings performed independently, independent of $(X_s)_{s \le t}$.

A few well-known properties:

Ergodic Markov chain, transition probabilities

$P(X_{t+1}=k \mid X_t=l) \;=$

$$\sum_{m=\max\{0,k+l-N\}}^{\min\{k,l\}} \binom{l}{m}\binom{N-l}{k-m}\,\alpha^m(1-\alpha)^{l-m}\,\beta^{k-m}(1-\beta)^{N-l+m-k},$$

uniquely determined stationary distribution: $\mathrm{Bin}(N,\pi)$.

Autocorrelation function: $\rho_X(k)=\rho^k$ for $k\geq 0$.

Regression properties:

$$E[X_{t+1}\mid X_t] \;=\; \rho\cdot X_t \;+\; N\beta,$$

$$V[X_{t+1}\mid X_t] \;=\; \rho(1-\rho)(1-2\pi)\cdot X_t \;+\; N\beta(1-\beta).$$

Limitations:

- Binomial marginal distribution,
- exponentially decaying ACF (AR(1)-like),
- thinning probabilities at time t

  do not depend on process up to that time, ...

Approaches for generalization:

- pth order dependence (AR(p)-like): Weiß (2009).
- State dependence of parameters π, ρ resp. α, β at time $t+1$
  - as function of X_t/N: "density dependence";
    (e.g., proportion of infectives or proportion of occupied patches)
  - acc. to X_t in which regime: "SET approach" (Tong).

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Density-Dependent Binomial AR(1) Processes

Definition & Properties

Definition: (Weiß & Pollett, 2014)

Let $\pi : [0;1] \to (0;1)$ and $\rho : [0;1] \to (0;1)$, so functions $\beta(y) := \pi(y)\,\big(1-\rho(y)\big)$ and $\alpha(y) := \beta(y)+\rho(y)$ also range $(0;1)$.

Write $\pi_{t+1} := \pi(X_t/N)$, $\rho_{t+1} := \rho(X_t/N)$,

and $\alpha_{t+1} := \alpha(X_t/N)$, $\beta_{t+1} := \beta(X_t/N)$.

DD-BAR(1) recursion

$$X_{t+1} \;=\; \alpha_{t+1} \circ X_t \;+\; \beta_{t+1} \circ (N - X_t),$$

thinnings performed independently, independent of $(X_s)_{s\leq t}$.

General properties: (Weiß & Pollett, 2014)

- time-homogeneous finite-state Markov chain with truly positive transition probabilities

$$P(k|l) \;:=\; P(X_t = k \mid X_{t-1} = l) \;=\; \sum_{m=\max\{0,k+l-N\}}^{\min\{k,l\}}$$
$$\binom{l}{m}\binom{N-l}{k-m}\,\big(\alpha(l/N)\big)^m\big(1-\alpha(l/N)\big)^{l-m}\,\big(\beta(l/N)\big)^{k-m}\big(1-\beta(l/N)\big)^{N-l+m-k};$$

- regression properties

$$E[X_t \mid X_{t-1}] \;=\; \rho_t\, X_{t-1} + N\,\beta_t,$$
$$V[X_t \mid X_{t-1}] \;=\; \rho_t(1-\rho_t)\,(1-2\pi_t)\,X_{t-1} + N\,\beta_t(1-\beta_t);$$

- ...

General properties: (cont.)

- due to primitive transition matrix $\mathbf{P} = \big(P(k|l)_{k,l=0,\ldots,N}\big)$, ergodic with unique stationary distribution, obtained numerically from equation

$$\mathbf{P}\,\boldsymbol{p} = \boldsymbol{p} \qquad \text{(eigenvalue problem)}$$

  (also allows to evaluate stationary moments);

- second-order moments with time-lag h from $P(X_t = k, X_{t-h} = l)$ as entries of $\mathbf{P}^h\,\mathrm{diag}(\boldsymbol{p})$, etc.;
- . . .

Large–N Approximations: Define

$$f(x) \;:=\; \alpha(x)\,x+\beta(x)\,(1-x) \;=\; \rho(x)\,x+\beta(x),$$
$$v(x) \;:=\; \alpha(x)\,\big(1-\alpha(x)\big)\,x+\beta(x)\,\big(1-\beta(x)\big)\,(1-x),$$

so $E[X_t/N\,|\,X_{t-1}/N]=f(X_{t-1}/N)$, $\quad N\,V[X_t/N\,|\,X_{t-1}/N]=v(X_{t-1}/N)$.

Several large–N results, among others

- law of large numbers $\quad X_t/N \overset{P}{\underset{N\to\infty}{\to}} y_t$, where $y_{t+1}=f(y_t)$;
- central limit law for scaled fluctuations $\quad \sqrt{N}\,(X_t/N-y_t)$;
- if y^* stable fixed point of f, and $\kappa=f'(y^*)$, then

$$\mu_X \;\approx\; N\,y^*, \qquad \sigma_X^2 \;\approx\; N\,\frac{v(y^*)}{1-\kappa^2}, \qquad \rho_X(k) \;\approx\; \kappa^k.$$

(Buckley & Pollett, 2010; Weiß & Pollett, 2014)

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Density-Dependent Binomial AR(1) Processes

Special Cases

Binomial index of dispersion for r.v. with support $\{0, \ldots, N\}$, and with mean μ and variance σ^2:

$$I_{\mathsf{d}} \; := \; \frac{N\sigma^2}{\mu(N-\mu)} \quad \in \; (0;\infty).$$

For Bin(N,π)-distributed r.v., we have $I_{\mathsf{d}} = 1$ for any $\pi \in (0;1)$.

If $I_{\mathsf{d}} > 1$, then **overdispersion** w.r.t. binomial distribution ("extra-binomial variation"),
while **underdispersion** refers to $I_{\mathsf{d}} < 1$.

Special Case 1: Binomial Over- or Underdispersion

Definition: Let $\rho(y) = \rho \in (0;1)$ be constant,

and $\pi(y) = a + by$ be linear, where $a, a+b \in (0;1)$. So

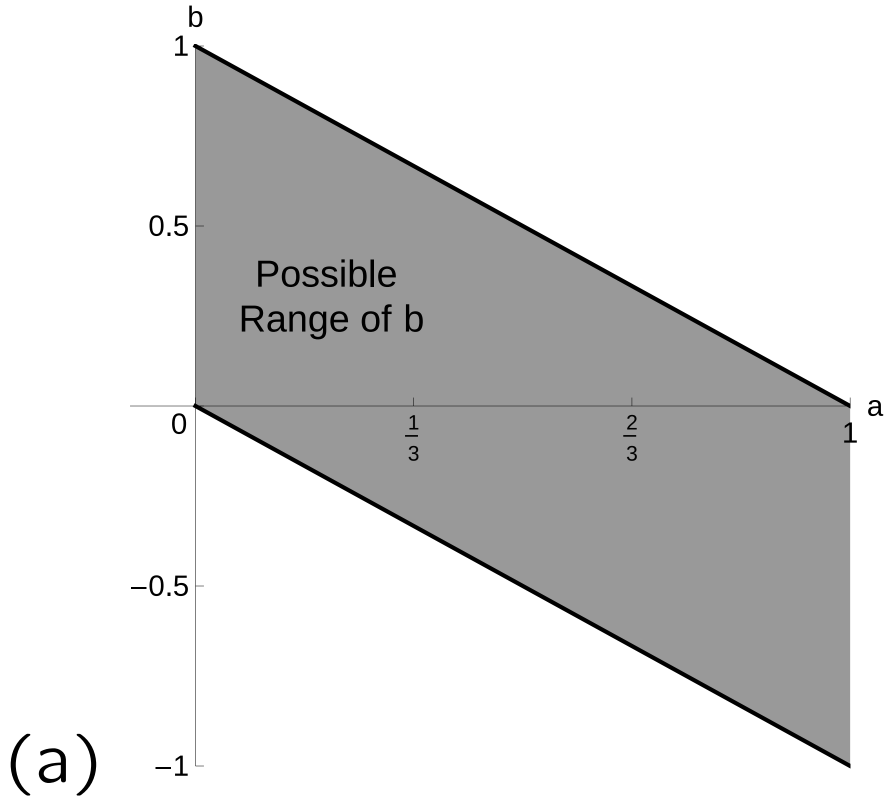
$$\beta_t = (1-\rho)\,(a + b\,X_{t-1}/N), \qquad \alpha_t = (1-\rho)\,(a + b\,X_{t-1}/N) + \rho.$$

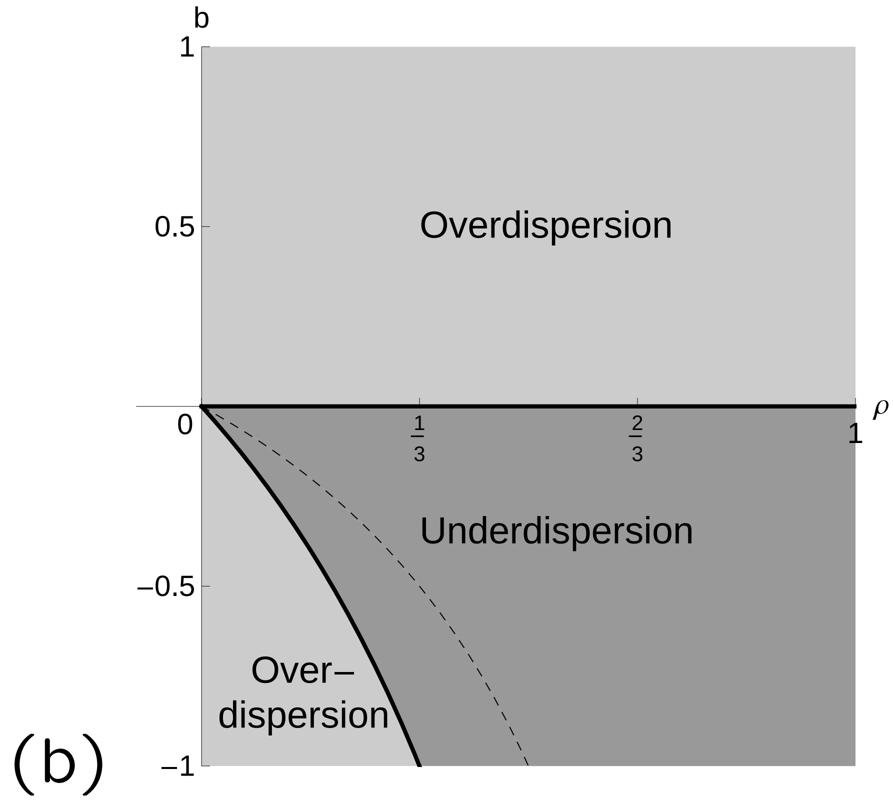
also linear, and depending on sign of b, these probabilities increase or decrease with increasing density.

Properties: ACF $\rho_X(k) \;=\; \big(\rho + (1-\rho)b\big)^k$, and

$$\mu \;=\; \frac{Na}{1-b}, \qquad I_{\mathrm{d}} \;=\; \frac{1-\rho^2}{1-\frac{\rho^2}{N}-(1-\frac{1}{N})\big(\rho+(1-\rho)b\big)^2}.$$

Properties:





Attainable range of b depending on a in (a)

dispersion determined by b and ρ in (b).

Data example 1: securities counts, $N = 22$.

Analyzing empirical (partial) autocorrelations,

first-order autoregressive dependence structure apparent.

Sample mean and variance are $\bar{x} \approx 9.529$ and $s^2 \approx 4.253$.

Empirical index of dispersion $\widehat{I}_{\mathsf{d}} = s^2/\big(\bar{x}(1-\bar{x}/N)\big) \approx 0.787$

indicates slight degree of binomial underdispersion.

ML-fitted model:

$$\widehat{a}_{\mathsf{ML}} \approx 0.693, \qquad \widehat{b}_{\mathsf{ML}} \approx -0.619, \qquad \widehat{\rho}_{\mathsf{ML}} \approx 0.630.$$

Boundary case $\rho \to 0$ leads to

$$X_t \overset{\mathsf{D}}{=} \mathsf{Bin}\big(N,\ a + b\,X_{t-1}/N\big).$$

Motivated by analogy to (Poisson) INARCH(1) model (Ferland et al., 2006): **binomial INARCH(1) model** (BINARCH(1)).

Properties: ACF $\rho_X(k) = b^k$, and

$$\mu = \frac{Na}{1-b}, \qquad I_{\mathsf{d}} = \frac{1}{1-(1-\frac{1}{N})b^2} \in [1; N).$$

Consequently, only overdispersion is possible.

Data example 2: hanta counts, $N = 38$.

Sample mean and variance are $\bar{x} \approx 4.173$ and $s^2 \approx 7.793$.

Empirical index of dispersion $\hat{I}_{\text{d}} \approx 2.098$

indicates considerable degree of binomial overdispersion.

	$\hat{\pi}_{\text{ML}}$		$\hat{\rho}_{\text{ML}}$	AIC	BIC
BAR(1)	0.115		0.535	222.8	226.7
	(0.013)		(0.071)		

	$\hat{a}_{\text{ML}}$	$\hat{b}_{\text{ML}}$	$\hat{\rho}_{\text{ML}}$	AIC	BIC
DD-BAR(1)	0.030	0.748	0.000	213.4	219.2
	(0.016)	(0.143)	(0.367)		

	$\hat{a}_{\text{ML}}$	$\hat{b}_{\text{ML}}$		AIC	BIC
BINARCH(1)	0.030	0.748		211.4	215.3
	(0.011)	(0.108)			

Special Case 2: Density-Dependent Colonization

Definition: Let $\alpha(y) = \alpha \in (0;1)$ be constant,

and $\beta(y) = \alpha\,(a + by)$ be linear, where $a, a + b \in (0;1]$.

Epidemic context: If $b > 0$,

prob. for susceptible becoming infected increases

if number of infectives already large (infection is spreading),

while recovery from infection independent of other infectives.

Properties:

$E[X_t \mid X_{t-1}] \;=\; (\alpha - \beta_t)\,X_{t-1} + N\,\beta_t$ quadratic in X_{t-1},

$V[X_t \mid X_{t-1}]$ cubic polynomial in X_{t-1}.

Explicit large-N approximation for marginal moments and ACF.

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Self-Exciting Threshold Binomial AR(1) Processes

(jointly with T.A. Möller, M.E. Silva, M.G. Scotto, I. Pereira)

"Work in Progress"

Idea: Like in Monteiro et al. (2012), fix threshold value $0 \leq R < N$, which separates range into two regimes: lower regime $\{0, 1, 2, \ldots, R\}$, upper regime $\{R+1, R+2, \ldots, N\}$.

Depending if previous observation in lower regime ($X_{t-1} \leq R$) or in upper regime ($X_{t-1} > R$), parameters of BAR(1) recursion at time t chosen differently.

Note: Possible generalizations include

- more than two regimes;
- increased delay $X_{t-d} \leq R$ vs. $X_{t-d} > R$ with $d > 1$;
- more complex criteria, e.g., $\max\{X_{t-1}, \ldots, X_{t-d}\} \leq R$.

Definition: Define $\pi_i \in (0;1)$, $\rho_i \in \Big(\max\{-\frac{\pi_i}{1-\pi_i},\ -\frac{1-\pi_i}{\pi_i}\};\ 1\Big)$, and $\beta_i := \pi_i \cdot (1-\rho_i) \in (0;1)$ and $\alpha_i := \beta_i + \rho_i \in (0;1)$.

SET-BAR(1) process $\quad X_t \;=\; \phi_t \circ X_{t-1} \;+\; \eta_t \circ (N - X_{t-1}),$

where $\phi_t := \alpha_1 I_{t-1} + \alpha_2 (1 - I_{t-1})$ and $\eta_t := \beta_1 I_{t-1} + \beta_2 (1 - I_{t-1})$ with indicator $I_{t-1} := \mathbb{1}_{\{X_{t-1} \leq R\}}$.

Note: If $R = 0$, then α_1 no influence (can be chosen arbitrarily).
$\Rightarrow$ α_1 unidentifiable during parameter estimation.
Same issue with β_2 and $R = N-1$. Therefore, we set $\rho_1 = \rho_2$ for $R = 0, N-1$ ($\rightarrow$ LSET model from below).

Properties:

SET-BAR(1) model instance of

DD-BAR(1) models by Weiß & Pollett (2014)

$\Rightarrow$ adapt results concerning transition probabilities, ergodicity, existence of unique stationary marginal distribution $\boldsymbol{p}$.

Since finite range, always possible to compute $\boldsymbol{p}$ numerically by solving eigenvalue problem $\mathbf{P}\,\boldsymbol{p} = \boldsymbol{p}$.

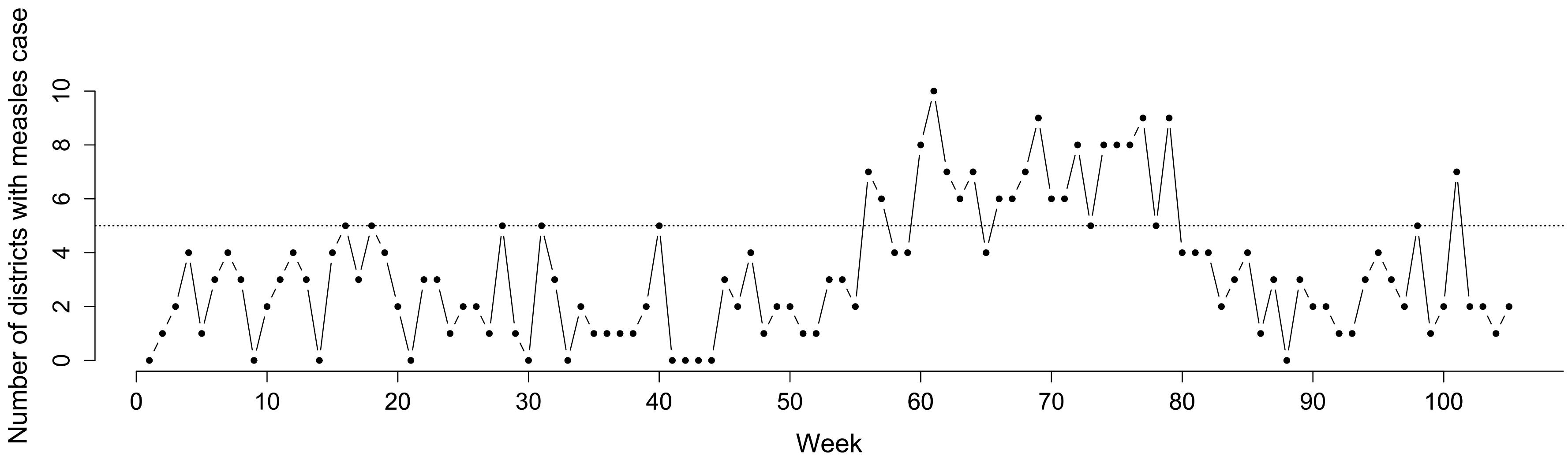
We derived closed-form formulae for mean and variance, but very complex.

Data example 3: (Robert-Koch-Institut)

Weekly No. of districts with new measles case (G., 2004/05).

Finite range $\{0, \dots, N\}$ with $N = 38$ (number of districts).

But . . .



. . . level shift due to measles epidemy!

Above measles time series: level shift in first half of 2005, but no obvious change in serial dependence structure.

Idea: Reduce number of parameters by additional restriction $\rho_1 = \rho_2 =: \rho$.

Definition: Let $\rho \in \Big(\max\{-\frac{\pi_1}{1-\pi_1},\ -\frac{\pi_2}{1-\pi_2},\ -\frac{1-\pi_1}{\pi_1},\ -\frac{1-\pi_2}{\pi_2}\};\ 1\Big)$.
A SET-BAR(1) process for which $\rho_1 = \rho_2 =: \rho \neq 0$ holds is called an **LSET-BAR(1) process**.

Defining $p := P(X_t \leq R) = E[I_{t-1}]$, $\mu_{IX} := E[I_{t-1}X_{t-1}]$,

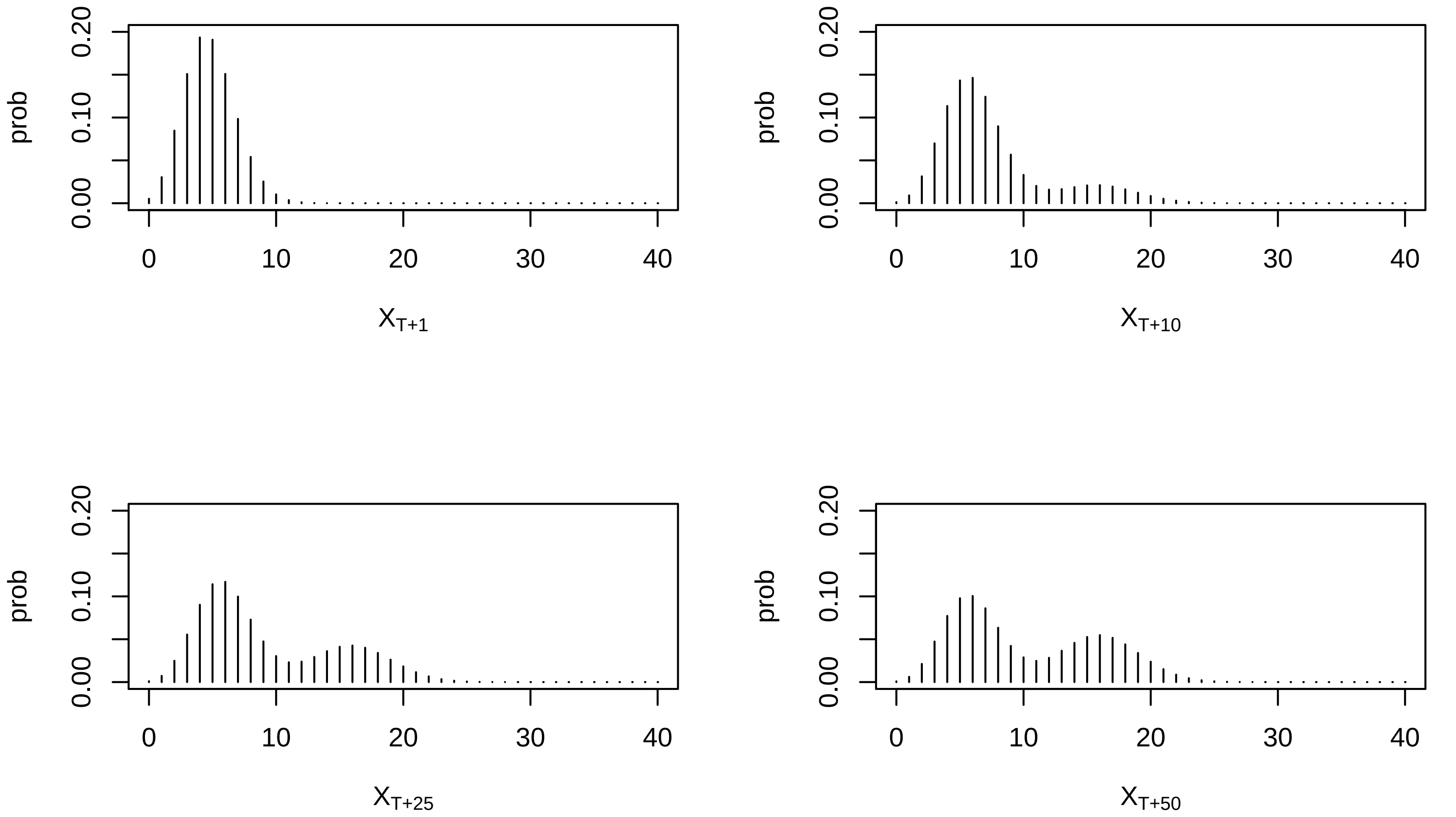
we can express unconditional mean and variance as

$$\begin{aligned} \mu_X &= Np\,\pi_1 \;+\; N(1-p)\,\pi_2, \\ \sigma_X^2 &= Np\,\pi_1(1-\pi_1) \;+\; N(1-p)\,\pi_2(1-\pi_2) \\ &+\; N^2 p(1-p)\,(\pi_2-\pi_1)^2 \\ &+\; \frac{2\rho}{1+\rho}(N-1)\,(\pi_2-\pi_1)\big(Np\,\pi_1-\mu_{IX}\big). \end{aligned}$$

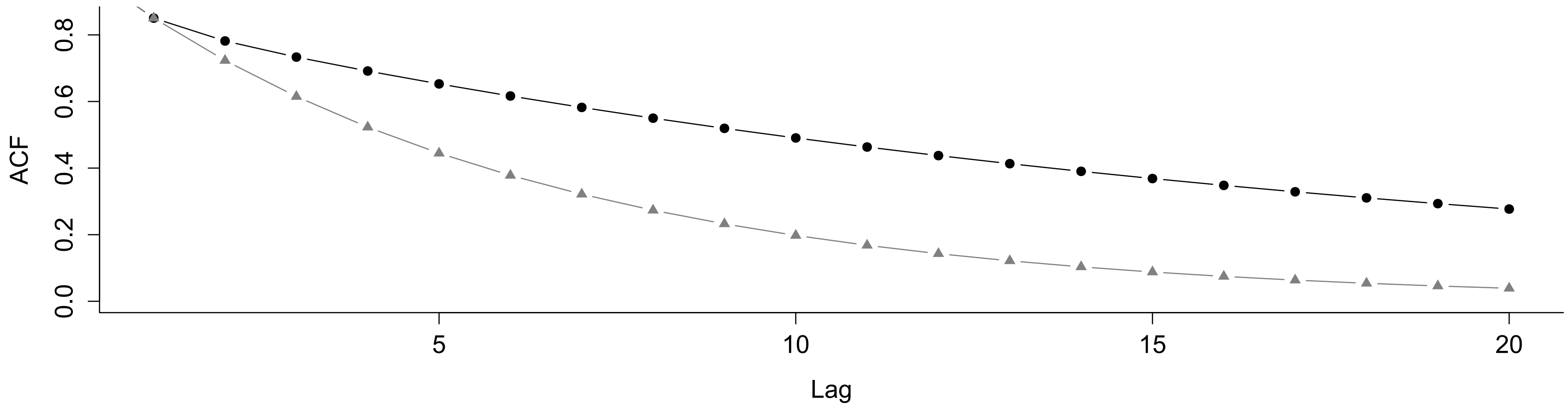
LSET model able to show over- and underdispersion

w.r.t. binomial distribution for appropriate parameter settings.

Forecasting distributions for model $N = 40$, $\rho = 0.3$, $\pi_1 = 0.15$, $\pi_2 = 0.4$ and $R = 10$, conditional on $X_T = 2$:



Autocorrelation function for model $N = 40$, $\rho = 0.3$, $\pi_1 = 0.15$, $\pi_2 = 0.4$ and $R = 10$ (black points):



Gray triangles show $f(k) := \big(\rho_X(1)\big)^k$

$\Rightarrow$ longer memory than corresponding AR(1)-like model.

Data example 3: measles counts, $N = 38$.

	Par. 1	Par. 2	Par. 3	Par. 4	AIC	BIC
BAR(1)	0.0882	0.4158	-	-	448.2	453.5
(π, ρ)	(0.0070)	(0.0550)				
DD-BAR(1)	0.0419	0.5270	0	-	436.9	444.9
(a, b, ρ)	(0.0095)	(0.1077)	(0.1765)			
BINARCH(1)	0.0419	0.5270	-	-	434.9	440.2
(a, b)	(0.0060)	(0.0682)				
SET-BAR(1)	0.0706	0.1558	0.1916	0.2904	432.5	443.1
$(\pi_1, \pi_2, \rho_1, \rho_2)$	(0.0056)	(0.0269)	(0.0884)	(0.375)		
LSET-BAR(1)	0.0707	0.1604	0.1947	-	430.6	438.5
(π_1, π_2, ρ)	(0.0057)	(0.0169)	(0.0876)			

All threshold models include threshold value $R = 5$.

Conclusions

- Time series of counts with finite range $\{0, 1, \ldots, N\}$ important topic in practice.
- McKenzie's basic binomial AR(1) model easily generalized in several ways.
- Density-dependent binomial AR(1) model for binomial over- or underdispersion, boundary case of binomial INARCH(1) model.
- Model with density-dependent colonization.
- SET models for counts exhibiting piecewise-type patterns.

Thank You

for Your Interest!

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