Bias Corrections for Moment Estimators in Poisson INAR(1) and INARCH(1) Processes

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Bias of Moment Estimators for Time Series Data

Background & Outline

Bias of Moment Estimators for Time Series

Moment estimators applied to time series $X_1, \ldots, X_T$ often biased parameter estimates, especially if time series short.

Lot of work for continuous-valued time series, especially (Gaussian) AR(1) model $X_t = \alpha \cdot X_{t-1} + \varepsilon_t$, e.g.:

- Kendall (1954): asympt. mean of $\widehat{\alpha}_{\mathsf{MM}}$ as $\alpha - 1/T\,(1 + 4\alpha)$, i.e., negative bias, increases linearly with α.
- Comprehensive work (including higher-order models) by Shaman & Stine (1988).
- Engle et al. (1985), ARCH(1), $\widehat{\alpha}_{\mathsf{ML}}$: asympt. mean $\alpha - 1/T\,(1 + 12\alpha)$, stronger bias than AR(1).

Count data time series:

Two equally popular counterparts to AR(1) model,

- **Poisson INAR(1) model** by McKenzie (1985),

  $X_t \;=\; \alpha \circ X_{t-1} \;+\; \epsilon_t, \quad \epsilon_t \sim \text{Poi}(\beta);$
- **Poisson INARCH(1) model** (e.g., Ferland et al. (2006)),

  $X_t \,\big|\, X_{t-1}, X_{t-2} \ldots \;\sim\; \text{Poi}(\beta + \alpha \cdot X_{t-1}).$

Both have mean $\mu = \beta/(1-\alpha)$, AR(1)-like ACF $\rho(k) = \alpha^k$,

but differ otherwise, especially variance:

- P. INAR(1): $\sigma^2 = \beta/(1-\alpha) \;= \mu$ (equidispersion);
- P. INARCH(1): $\sigma^2 = \beta/\big((1-\alpha)(1-\alpha^2)\big) \;> \mu$ (overdisp.).

Count data time series: (cont.)

Topic of **bias correction** only little attention, **exceptions:**

- Jung et al. (2005): simulation study of estimators for α in Poisson INAR(1), "unfavourable bias properties". Improvement via Gaussian AR(1) correction.
- Bourguignon & Vasconcellos (2014): squared difference estimator of β for Poisson INAR(1).
- Weiß & Schweer (2015): dispersion ratio for Poisson INAR(1) and INARCH(1).

Aim of present talk:

Comprehensive analysis of

asymptotic distributions and bias corrections of

common moment estimators for P. INAR(1) and INARCH(1):

- empirical mean $\bar{X}$, variance $\widehat{\gamma}(0)$ and ACV $\widehat{\gamma}(1)$;
- $\widehat{\rho}(1) = \widehat{\gamma}(1)/\widehat{\gamma}(0)$ (moment estimator of α) and $\bar{X}\big(1-\widehat{\rho}(1)\big)$ (moment estimator of β).

Starting with two central limit theorems, we derive

explicit approximations for bias and standard deviation.

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Two Central Limit Theorems

Technical prerequisites

Sketch: Vector-valued process

$$\boldsymbol{Y}_t := \big(X_t-\mu,\ X_t^2-\mu(0),\ X_tX_{t-1}-\mu(1)\big)^\top \text{ with } \mu(k) := E[X_tX_{t-k}],$$

then . . .

Theorem 1: Let $(X_t)_{\mathbb{Z}}$ be Poisson INAR(1), then

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\boldsymbol{Y}_t \xrightarrow{\mathcal{D}} \mathrm{N}\,(\mathbf{0}, \boldsymbol{\Sigma}) \qquad \text{with } \boldsymbol{\Sigma} = \big(\sigma_{ij}\big) \text{ given by } \ldots$$

Theorem 2: Let $(X_t)_{\mathbb{Z}}$ be Poisson INARCH(1), then

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\boldsymbol{Y}_t \xrightarrow{\mathcal{D}} \mathrm{N}\,(\mathbf{0}, \boldsymbol{\Sigma}) \qquad \text{with } \boldsymbol{\Sigma} = \big(\sigma_{ij}\big) \text{ given by } \ldots$$

Sketch: (cont.)

Explicit formulae for σ_{ij} together with proofs in paper.

Then asymptotic normality and standard deviations
for above estimators via Delta theorem,

explicit bias approximation via second-order Taylor expansion.

Again, proofs in paper.

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Moment Estimation for Poisson INAR(1) Model

Results & Applications

Moment Estimation for Poisson INAR(1) Model

Theorem 3: Let $(X_t)_{\mathbb{Z}}$ be Poisson INAR(1), then $\big(\bar{X}, \hat{\gamma}(0), \hat{\gamma}(1)\big)^\top$ asymptotically normal with mean vector

$$\left(\mu, \quad \mu - \frac{1}{T}\frac{1+\alpha}{1-\alpha}\mu, \quad \mu\,\alpha - \frac{1}{T}\frac{1+\alpha}{1-\alpha}\mu\right)^\top$$

and covariance matrix

$$\frac{1}{T}\begin{pmatrix} \frac{1+\alpha}{1-\alpha}\mu & \frac{1+\alpha}{1-\alpha}\mu & \frac{2\alpha}{1-\alpha}\mu \\ * & 2\,\frac{1+\alpha^2}{1-\alpha^2}\mu^2 + \frac{1+\alpha}{1-\alpha}\mu & \frac{4\alpha}{1-\alpha^2}\mu^2 + \frac{2\alpha}{1-\alpha}\mu \\ * & * & \frac{1+4\alpha^2-\alpha^4}{1-\alpha^2}\mu^2 + \frac{\alpha(1+\alpha)}{1-\alpha}\mu \end{pmatrix}.$$

Theorem 4: Let $(X_t)_{\mathbb{Z}}$ be Poisson INAR(1), then $\Big(\hat{\rho}(1),\ \bar{X}\big(1-\hat{\rho}(1)\big)\Big)^\top$ asymptotically normal with mean vector

$$\left(\alpha - \frac{1}{T}\left(1+3\alpha+\frac{\alpha}{\beta}(1-\alpha)\right),\quad \beta + \frac{1}{T}\,\frac{1+3\alpha}{1-\alpha}\,\beta\right)^\top$$

and covariance matrix

$$\frac{1}{T}\begin{pmatrix} 1-\alpha^2+\frac{\alpha(1-\alpha)^2}{\beta} & -(1+\alpha)\,\beta \\ -(1+\alpha)\,\beta & \beta+\frac{1+\alpha}{1-\alpha}\,\beta^2 \end{pmatrix}.$$

Note:

Bias of $\hat{\rho}(1)$ for Gaussian AR(1): $-1/T\,(1+4\alpha)$ (Kendall 1954).

Here, $-1/T\,\big(1+3\alpha+\alpha/\mu\big)$ similar, but further influenced by μ.

Poisson INAR(1), $\beta = 2$: asymptotic & simulated means.

		$\bar{X}$		$\widehat{\gamma}(0)$		$\widehat{\gamma}(1)$		$\widehat{\rho}(1)$		$\bar{X}\big(1-\widehat{\rho}(1)\big)$	
α	T	asy.	sim.	asy.	sim.	asy.	sim.	asy.	sim.	asy.	sim.
0.25	100	2.667	2.665	2.622	2.620	0.622	0.609	0.232	0.227	2.047	2.058
	250	2.667	2.666	2.649	2.653	0.649	0.646	0.243	0.241	2.019	2.022
	500	2.667	2.665	2.658	2.655	0.658	0.657	0.246	0.246	2.009	2.008
	1000	2.667	2.667	2.662	2.663	0.662	0.661	0.248	0.248	2.005	2.007
0.50	100	4.000	4.000	3.880	3.879	1.880	1.858	0.474	0.468	2.100	2.124
	250	4.000	3.999	3.952	3.951	1.952	1.946	0.490	0.488	2.040	2.046
	500	4.000	4.002	3.976	3.979	1.976	1.974	0.495	0.494	2.020	2.025
	1000	4.000	3.999	3.988	3.989	1.988	1.988	0.497	0.497	2.010	2.010
0.75	100	8.000	7.993	7.440	7.451	5.440	5.391	0.717	0.708	2.260	2.328
	250	8.000	7.992	7.776	7.768	5.776	5.746	0.737	0.734	2.104	2.126
	500	8.000	7.997	7.888	7.895	5.888	5.883	0.743	0.742	2.052	2.063
	1000	8.000	7.998	7.944	7.943	5.944	5.936	0.747	0.746	2.026	2.033

Approximated & simul. values show rather good agreement.

Moment Estimation for Poisson INAR(1) Model

Uncorrected moment estimates $\hat{\alpha}_{\mathsf{MM}}, \hat{\beta}_{\mathsf{MM}}$.

Application for bias correction:

$$\hat{\beta}_{\mathsf{MM;\,corr}} \;=\; \hat{\beta}_{\mathsf{MM}} \left(1+\frac{1}{T}\,\frac{1+3\hat{\alpha}_{\mathsf{MM;\,corr}}}{1-\hat{\alpha}_{\mathsf{MM;\,corr}}}\right)^{-1},$$

$\hat{\alpha}_{\mathsf{MM;\,corr}}$ as solution of quadratic equation

$$\hat{\alpha}^2_{\mathsf{MM;\,corr}}\cdot\tfrac{1}{T}\,\tfrac{1}{\hat{\beta}_{\mathsf{MM}}}\left(1-\tfrac{3}{T}\right) + \hat{\alpha}_{\mathsf{MM;\,corr}}\cdot\left(1-\tfrac{3}{T}-\tfrac{1}{T}\left(1+\tfrac{1}{T}\right)\tfrac{1}{\hat{\beta}_{\mathsf{MM}}}\right) - \hat{\alpha}_{\mathsf{MM}}-\tfrac{1}{T} \;=\; 0.$$

If $T \geq 4$, $\hat{\alpha}_{\mathsf{MM}} \in [0;1)$ and $\hat{\beta}_{\mathsf{MM}} > 0$, condition

$$\hat{\alpha}_{\mathsf{MM}} \;<\; 1-\tfrac{4}{T}\left(1+\tfrac{1}{T}\,\tfrac{1}{\hat{\beta}_{\mathsf{MM}}}\right)$$

guarantees unique solution in $(0;1)$ for $\hat{\alpha}_{\mathsf{MM;\,corr}}$.

Bias-corrected moment estimates $\hat{\alpha}_{\text{MM; corr}}, \hat{\beta}_{\text{MM; corr}}$:

T	$\hat{\alpha}_{\text{MM}} = 0.250$	$\hat{\beta}_{\text{MM}} = 2.000$	$\hat{\alpha}_{\text{MM}} = 0.500$	$\hat{\beta}_{\text{MM}} = 2.000$	$\hat{\alpha}_{\text{MM}} = 0.750$	$\hat{\beta}_{\text{MM}} = 2.000$
100	0.269	1.952	0.527	1.896	0.785	1.731
250	0.257	1.981	0.511	1.959	0.764	1.895
500	0.254	1.991	0.505	1.980	0.757	1.948
1000	0.252	1.995	0.503	1.990	0.753	1.974

Visible effect of bias correction even for $T = 500$.

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Moment Estimation for Poisson INARCH(1) Model

Results & Applications

Theorem 5: Let $(X_t)_{\mathbb{Z}}$ be Poisson INARCH(1), denote $f_k := \beta / \prod_{i=1}^{k}(1-\alpha^i)$, then $\big(\bar{X}, \hat{\gamma}(0), \hat{\gamma}(1)\big)^\top$ asymptotically normal with mean vector

$$\left(f_1,\quad f_2 - \frac{1}{T}\,\frac{f_1}{(1-\alpha)^2},\quad f_2\,\alpha - \frac{1}{T}\,\frac{f_1}{(1-\alpha)^2}\right)^\top$$

and covariance matrix

$$\frac{1}{T}\begin{pmatrix} \frac{f_1}{(1-\alpha)^2} & \frac{1+\alpha+2\alpha^2}{(1-\alpha)(1-\alpha^2)}\,f_2 & \frac{\alpha(2+\alpha+\alpha^2)}{(1-\alpha)(1-\alpha^2)}\,f_2 \\ * & 2\,\frac{1+\alpha^2}{1-\alpha^2}\,f_2^2 + \frac{1+2\alpha+8\alpha^2+9\alpha^3+4\alpha^4+6\alpha^5}{(1-\alpha^2)^2}\,f_3 & \frac{4\alpha}{1-\alpha^2}\,f_2^2 + \frac{\alpha(2+5\alpha+8\alpha^2+10\alpha^3+3\alpha^4+2\alpha^5)}{(1-\alpha^2)^2}\,f_3 \\ * & * & \frac{1+4\alpha^2-\alpha^4}{1-\alpha^2}\,f_2^2 + \frac{\alpha(1+3\alpha+8\alpha^2+8\alpha^3+8\alpha^4+2\alpha^5)}{(1-\alpha^2)^2}\,f_3 \end{pmatrix}.$$

Theorem 6: Let $(X_t)_{\mathbb{Z}}$ be Poisson INARCH(1), then $\Big(\widehat{\rho}(1),\ \bar{X}\big(1-\widehat{\rho}(1)\big)\Big)^\top$ asymptotically normal with mean vector

$$\left(\alpha - \frac{1}{T}\left(1+3\alpha+\frac{\alpha}{\beta}\left(1+\frac{2\alpha(1+2\alpha^2)}{1+\alpha+\alpha^2}\right)\right),\quad \beta+\frac{1}{T}\left(\frac{1+3\alpha}{1-\alpha}\,\beta+\frac{2\alpha^2(1+2\alpha^2)}{1-\alpha^3}\right)\right)^\top$$

and covariance matrix

$$\frac{1}{T}\begin{pmatrix} 1-\alpha^2+\frac{\alpha(1-\alpha^2)(1+2\alpha^2)}{\beta(1+\alpha+\alpha^2)} & -(1+\alpha)\,\beta-\frac{(1+2\alpha)\alpha^3}{1+\alpha+\alpha^2} \\ -(1+\alpha)\,\beta-\frac{(1+2\alpha)\alpha^3}{1+\alpha+\alpha^2} & \frac{1+2\alpha^4}{1-\alpha^3}\,\beta+\frac{1+\alpha}{1-\alpha}\,\beta^2 \end{pmatrix}.$$

Note: INARCH(1) case with stronger bias for $\widehat{\rho}(1)$, with

$1-\alpha$ vs. $1+2\alpha(1+2\alpha^2)/(1+\alpha+\alpha^2)$,

for $\bar{X}\big(1-\widehat{\rho}(1)\big)$, with additional term $2\alpha^2(1+2\alpha^2)/(1-\alpha^3)$.

Poisson INARCH(1), $\beta = 2$: asymptotic & simulated means.

		$\bar{X}$		$\widehat{\gamma}(0)$		$\widehat{\gamma}(1)$		$\widehat{\rho}(1)$		$\bar{X}\big(1-\widehat{\rho}(1)\big)$	
α	T	asy.	sim.	asy.	sim.	asy.	sim.	asy.	sim.	asy.	sim.
0.25	100	2.667	2.667	2.797	2.799	0.664	0.657	0.231	0.228	2.048	2.055
	250	2.667	2.668	2.825	2.828	0.692	0.691	0.242	0.242	2.019	2.022
	500	2.667	2.666	2.835	2.835	0.702	0.701	0.246	0.246	2.010	2.010
	1000	2.667	2.666	2.840	2.840	0.706	0.706	0.248	0.248	2.005	2.005
0.50	100	4.000	4.003	5.173	5.160	2.507	2.469	0.470	0.465	2.109	2.132
	250	4.000	3.998	5.269	5.263	2.603	2.586	0.488	0.486	2.043	2.051
	500	4.000	4.000	5.301	5.304	2.635	2.633	0.494	0.494	2.022	2.024
	1000	4.000	4.001	5.317	5.324	2.651	2.652	0.497	0.497	2.011	2.013
0.75	100	8.000	8.012	17.006	17.157	12.434	12.435	0.709	0.704	2.301	2.345
	250	8.000	8.011	17.774	17.874	13.202	13.245	0.733	0.732	2.121	2.134
	500	8.000	7.996	18.030	18.050	13.458	13.451	0.742	0.741	2.060	2.069
	1000	8.000	8.001	18.158	18.144	13.586	13.564	0.746	0.745	2.030	2.036

Approximated & simul. values show rather good agreement.

Applications:

Claim counts ($T = 96$) discussed in Weiß (2009).

- ML estimates (INARCH(1)): $\hat{\alpha}_{\mathrm{ML}} \approx 0.483$, $\hat{\beta}_{\mathrm{ML}} \approx 4.486$
- Uncorrected MM: $\hat{\alpha}_{\mathrm{MM}} \approx 0.452$, $\hat{\beta}_{\mathrm{MM}} \approx 4.711$
- Bias-corrected MM: $\hat{\alpha}_{\mathrm{MM;\,corr}} \approx 0.480$, $\hat{\beta}_{\mathrm{MM;\,corr}} \approx 4.484$.

Strikes counts ($T = 108$) discussed in Weiß (2010).

- ML estimates (INARCH(1)): $\hat{\alpha}_{\mathrm{ML}} \approx 0.636$, $\hat{\beta}_{\mathrm{ML}} \approx 1.811$
- Uncorrected MM: $\hat{\alpha}_{\mathrm{MM}} \approx 0.573$, $\hat{\beta}_{\mathrm{MM}} \approx 2.109$
- Bias-corrected MM: $\hat{\alpha}_{\mathrm{MM;\,corr}} \approx 0.605$, $\hat{\beta}_{\mathrm{MM;\,corr}} \approx 1.964$.

Conclusions

- Explicit expressions for asymptotic bias of diverse moment estimators under Poisson INAR(1) or INARCH(1) process.
- Good finite-sample performance.
- Feasible way to obtain bias-corrected moment estimates.
- Successfully applied to real-data examples.

Thank You

for Your Interest!

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