Newsvendor Model in Presence of Correlated Discrete Demand

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The Classic Newsvendor Model

Background & Example

Single-period inventory management:

uncertain demand,

determine optimal ordering quantity for next period,

perishable goods (i.e., unsold goods cannot be carried from one period to the next, e.g., newspaper)

$\rightarrow$ „newsvendor model".

- Ordering quantity too small (unsatisfied demand): „underage costs" c_u per missing unit.
- Ordering quantity too large (leftover inventory): „overage costs" c_o per surplus unit.

Here, **demand** process $(D_t)_{\mathbb{N}_0}$ is **count data** process, e.g.,

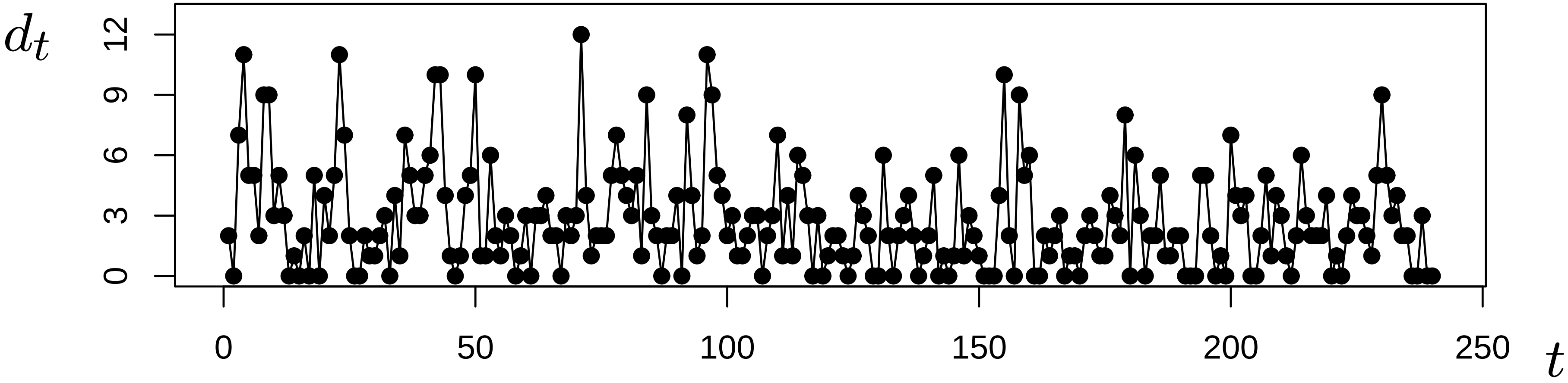
daily number of blood bags (red blood cells) for hospital.

Typical costs per bag: \$20-\$40.

High extra cost per missing bag: \$80-\$100.

(„emergency delivery“)

Available data set (June 2009 to Jan. 2010, $T = 240$):



Marginal approach, stationary marginal distr. F of $(D_t)_{\mathbb{N}_0}$:

Fixed ordering quantity Q per period, resulting costs

$$C(Q, D_t) \;=\; \begin{cases} c_o\,(Q - D_t) & \text{if } D_t < Q, \\ c_u\,(D_t - Q) & \text{if } D_t > Q. \end{cases}$$

Expected costs

$$E\big[C(Q, D_t)\big] \;=\; E\Big[\max\big\{c_o\,(Q - D_t),\; c_u\,(D_t - Q)\big\}\Big]$$

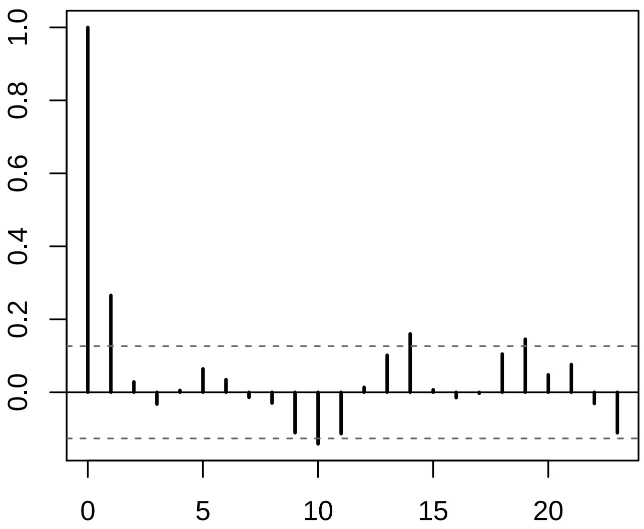
become minimal iff **optimal ordering quantity**

$$Q^* \;=\; \min\big\{Q \in \mathbb{N}_0 \mid F(Q) \geq \beta\big\} \quad \text{with „critical ratio“ } \beta = \frac{c_u}{c_o + c_u}.$$

Back to **data example** of blood bags:

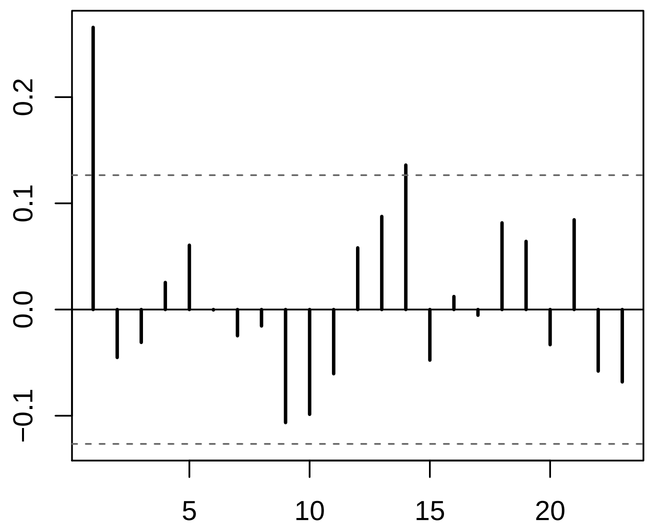
Looking at sample ACF and PACF,

$\widehat{\rho}_D(k)$



k

$\widehat{\rho}_{D;\,\mathrm{part}}(k)$



k

significant autocorrelation visible, AR(1)-like.

Question: Possible to improve ordering strategy by utilizing serial dependence between demand counts?

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Modeling Correlated Discrete Demand

INAR(1) Model

Blood bags data example: AR(1)-like autocorrelation.

⇒ **1st idea:** Apply basic Gaussian AR(1) model,

$$D_t = \alpha \cdot D_{t-1} + \epsilon_t, \qquad |\alpha| \leq 1.$$

But does not preserve discreteness of demand counts.

⇒ **2nd idea:** Apply **INAR(1) model** (McKenzie, 1985),

$$D_t = \alpha \circ D_{t-1} + \epsilon_t,$$

where '$\circ$' is **binomial thinning** operator, i.e.,

$\alpha \circ D \ \sim\ \text{Bin}(D, \alpha)$ (Steutel & van Harn, 1979).

Basics about INAR(1) processes:

- Intuitive interpretation:

$$\underbrace{D_t}_{\text{Population at time } t} = \underbrace{\alpha \circ D_{t-1}}_{\text{Survivors of time } t-1} + \underbrace{\epsilon_t}_{\text{Immigration}}.$$

- Homogeneous Markov chain, stationary solution satisfies

$$\mu_D = \frac{\mu_\epsilon}{1-\alpha}, \qquad \frac{\sigma_D^2}{\mu_D} = \frac{\frac{\sigma_\epsilon^2}{\mu_\epsilon} + \alpha}{1+\alpha}.$$

- Autocorrelation function $\rho_D(k) = \alpha^k$, i.e., AR(1)-type.
- One-step ahead transition probabilities:

$$P(D_{t+1} = k | D_t = l) = \sum_{j=0}^{\min\{k,l\}} \binom{l}{j} \alpha^j (1-\alpha)^{l-j} \cdot P(\epsilon_t = k-j).$$

INAR(1) Model for Correlated Discrete Demand

Features of marginal distribution of observed demands D_t

determined through

corresponding properties of innovations' distribution, e.g.,

- $\epsilon_t \sim \text{Poi}(\lambda) \quad \Rightarrow \quad D_t \sim \text{Poi}(\frac{\lambda}{1-\alpha})$,

  Poisson INAR(1) model, equidispersion;

- $\epsilon_t \sim \text{NB}(n, \pi)$ (Schweer & Weiß, 2014),

  NB INAR(1) model, overdispersion;

- $\epsilon_t \sim \text{ZIP}(\lambda, \omega)$ (Jazi et al., 2012),

  ZIP INAR(1) model, zero inflation.

Blood bags data example:

Estimated mean and variance are $\bar{d} \approx 2.792$ and $s_D^2/\bar{d} \approx 2.44$, i.e., strong degree of overdispersion. ACF $\hat{\rho}_D(1) \approx 0.266$.

Model	ML estimates of Par. 1	Par. 2	Par. 3	AIC	BIC	Fitted model: μ_D	σ_D^2/μ_D
iid Poi (λ)	2.792 (0.108)			1157.9	1161.3	2.792	1
Poi INAR(1) (λ, α)	2.322 (0.134)	0.167 (0.036)		1135.8	1142.8	2.789	1
iid NB (n, π)	1.831 (0.306)	0.396 (0.043)		1039.2	1046.2	2.792	2.53
NB INAR(1) (n, π, α)	1.370 (0.279)	0.374 (0.046)	0.179 (0.048)	1029.1	1039.6	2.789	2.67

$\Rightarrow$ NB INAR(1) model best choice for blood bags data.

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INAR(1) Implementation of Newsvendor Problem

Conditional approach

Idea behind INAR(1) implementation:

Utilize serial dependence by determining order quantity Q

for period $t+1$ based on previous demand D_t,

i.e., $Q_{t+1} = Q(D_t)$ varies with time t.

Expected cost $E\big[C\big(Q(D_t), D_{t+1}\big)\big]$

depend on joint distribution of (D_t, D_{t+1}).

Monotonicity of expectation op. plus law of total expectation:

expected cost $E\big[C\big(Q(D_t), D_{t+1}\big)\big]$ minimized

if Q^*_{t+1} minimizes

conditional expected cost $E\big[C\big(Q(D_t), D_{t+1}\big) \mid D_t\big]$.

Remember: marginal approach

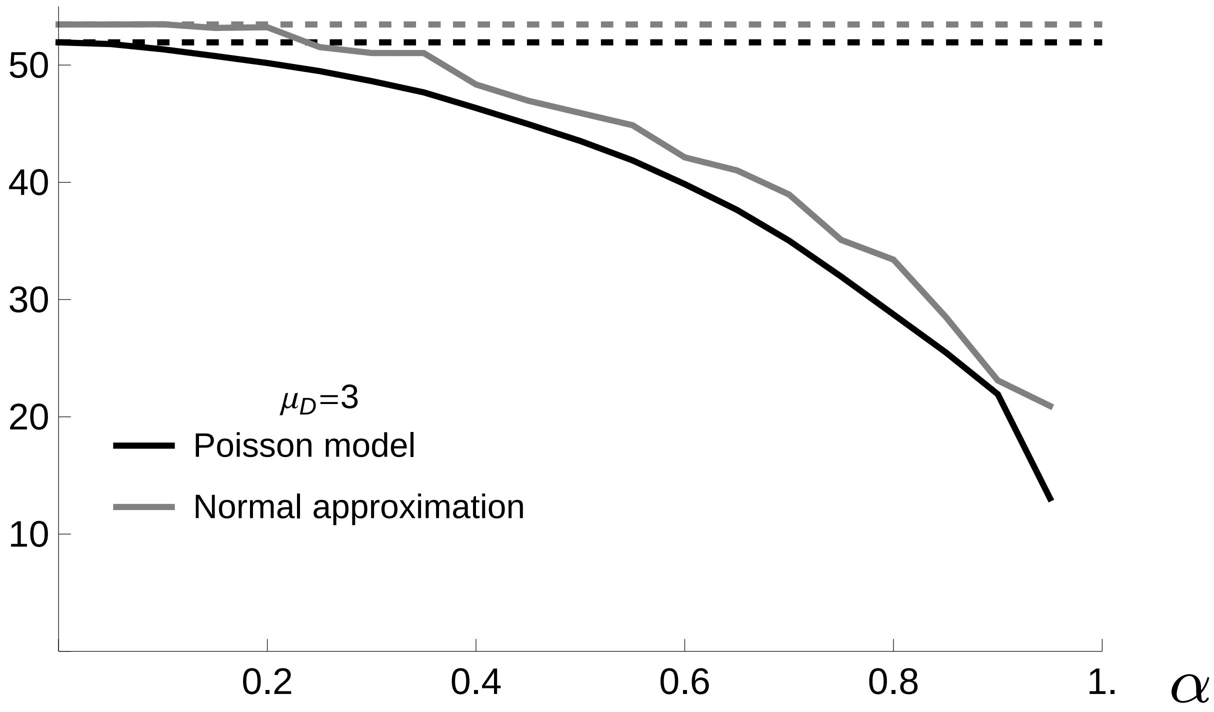
$$Q^* = \min\left\{Q \in \mathbb{N}_0 \mid F(Q) \geq \beta\right\} \quad \text{with „critical ratio“ } \beta = \frac{c_u}{c_o + c_u}.$$

Conditional approach:

Let $F_{D_{t+1}|D_t=l}(k)$ be conditional CDF of INAR(1) process, define order quantity

$$Q^*_{t+1}(D_t = l) = \min\left\{Q \in \mathbb{N}_0 \mid F_{D_{t+1}|D_t=l}(Q) \geq \beta\right\}.$$

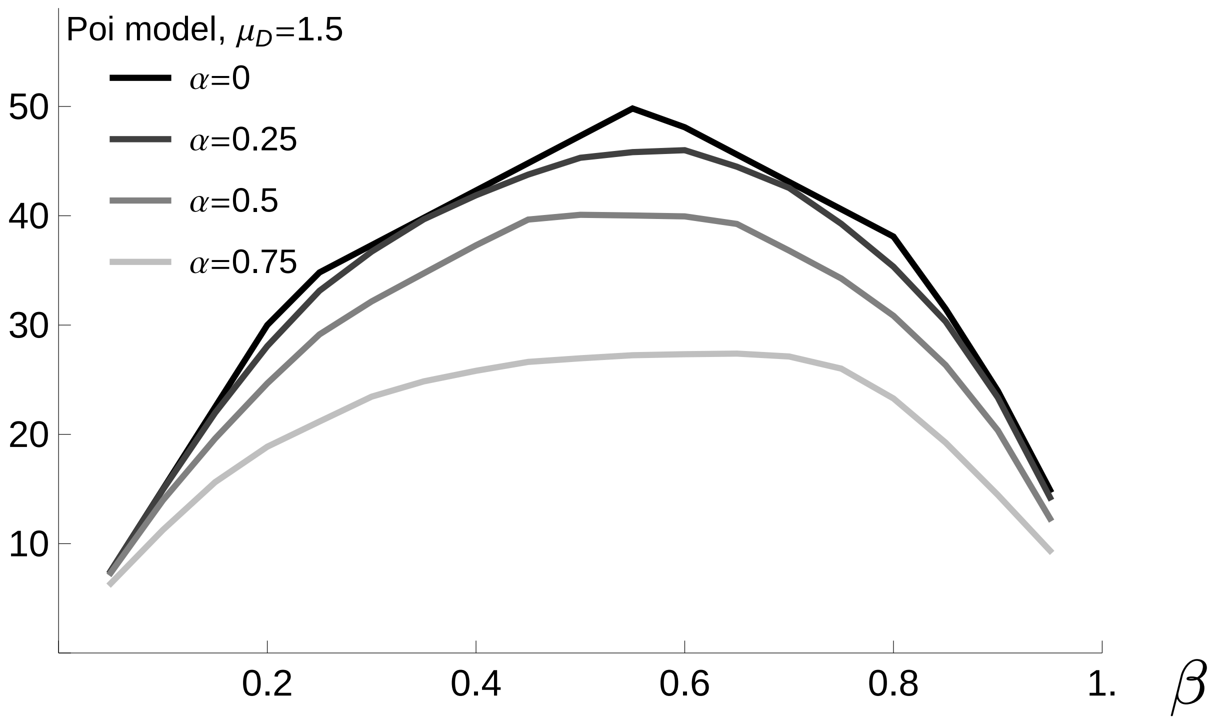
Expected cost against α, cost model \$80–\$20 ($\beta = 0.80$), Poi INAR(1) model with $\mu_D = 3$ (and corr. normal approximation):



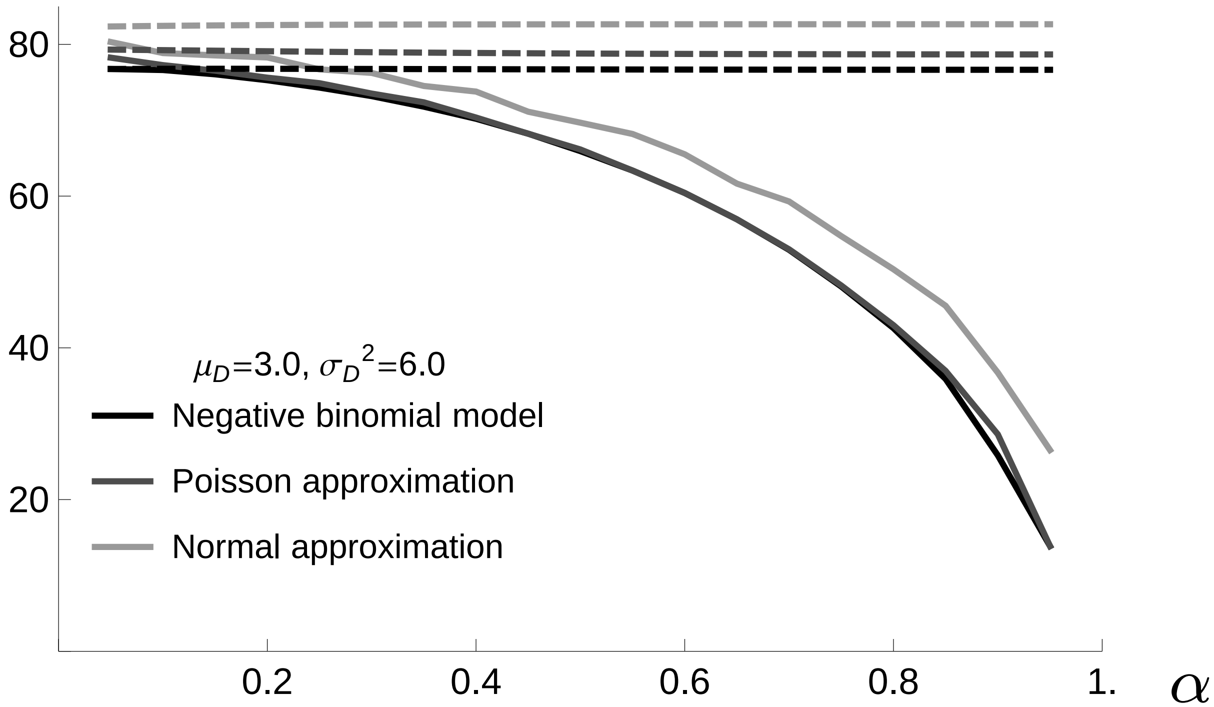
Expected cost against β,

cost model $c_u = \beta \cdot \$100$ and $c_o = (1 - \beta) \cdot \$100$,

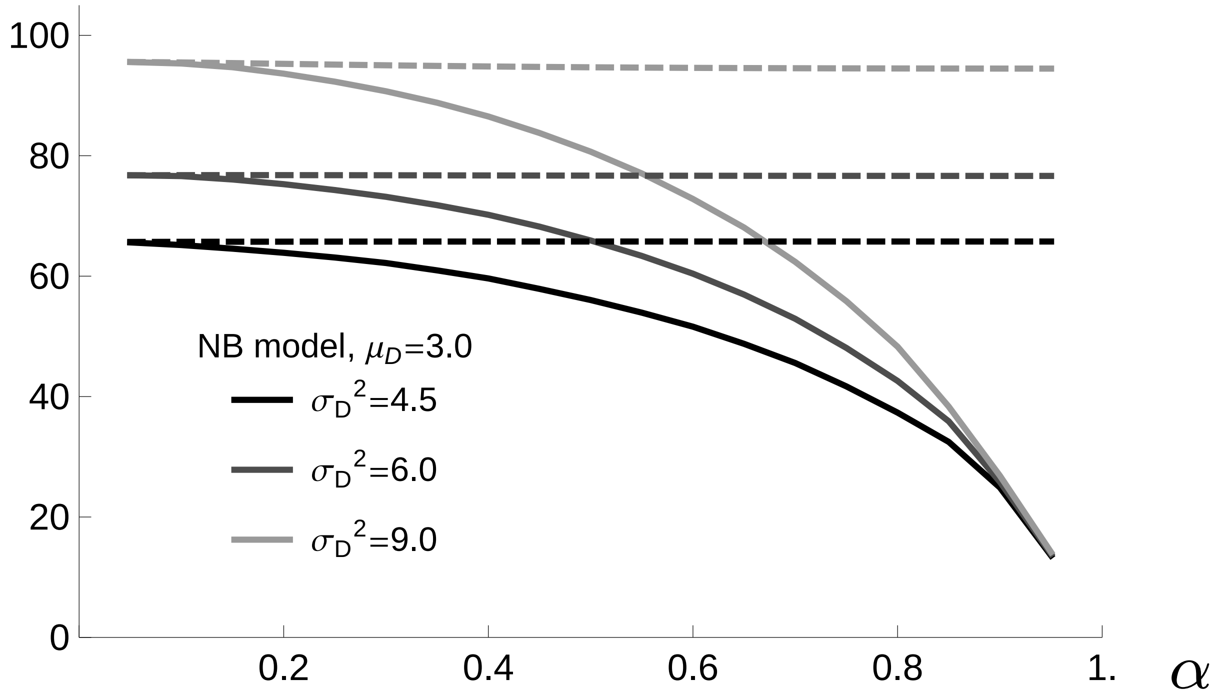
Poi INAR(1) model with $\mu_D = 1.5$ and different autocorr. levels:



Expected cost against α, cost model \$80–\$20 ($\beta = 0.80$), NB INAR(1) model with $\mu_D = 3$ and $\sigma_D^2 = 6$ (and corr. Poisson- resp. normal approximation):



Expected cost against α, cost model \$80–\$20 ($\beta = 0.80$), NB INAR(1) model with $\mu_D = 3$ and different dispersion ratios:



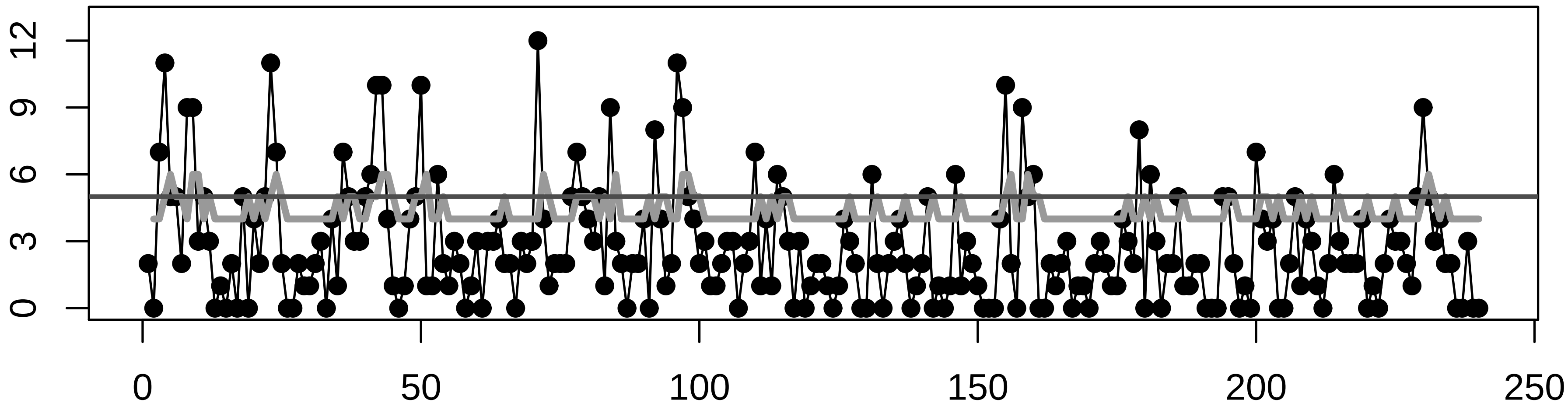
INAR(1) Implementation of Newsvendor Problem

Blood bags data: $(\bar{d} \approx 2.79,\ s_D^2 \approx 6.81,\ \hat{\rho}_D(1) \approx 0.27)$

Fitted NB INAR(1) model, cost model \$80–\$20,

gives

- marginal: $Q^* = 5$ and total cost \$20 040;
- conditional: total cost \$19 380.



Conclusions

- Novel implementation of newsvendor problem for autocorrelated discrete demands stemming from INAR(1) process.

- Comparison of dynamic INAR(1) implementation against traditional static implementation (also against approximating dynamic implementations).

- Dynamic INAR(1) implementation for appropriately specified conditional model always gives cost reduction, also much better than Gaussian AR(1) implementation.

Thank You

for Your Interest!

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