Newsvendor Model in Presence of Correlated Discrete Demand

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The Classic Newsvendor Model

Background & Example
The Classic Newsvendor Model

Single-period inventory management:

uncertain demand,
determine optimal ordering quantity for next period,
perishable goods (i.e., unsold goods cannot be carried from one
period to the next, e.g., newspaper)

→ „newsvendor model“.

- Ordering quantity too small (unsatisfied demand):
  „underage costs“ $c_u$ per missing unit.
- Ordering quantity too large (leftover inventory):
  „overage costs“ $c_o$ per surplus unit.

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Here, demand process \((D_t)_{t=0}^T\) is count data process, e.g., daily number of blood bags (red blood cells) for hospital. Typical costs per bag: $20-$40. High extra cost per missing bag: $80-$100. (“,emergency delivery“)

**Available data set** (June 2009 to Jan. 2010, \(T = 240\)): 

![Demand Data Chart]

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**Marginal approach**, stationary marginal distr. $F$ of $(D_t)_{N_0}$:

Fixed ordering quantity $Q$ per period, resulting costs

$$C(Q, D_t) = \begin{cases} c_o (Q - D_t) & \text{if } D_t < Q, \\ c_u (D_t - Q) & \text{if } D_t > Q. \end{cases}$$

Expected costs

$$E[C(Q, D_t)] = E \left[ \max \{c_o(Q - D_t), c_u(D_t - Q)\} \right]$$

become minimal iff **optimal ordering quantity**

$$Q^* = \min \{Q \in N_0 | F(Q) \geq \beta\} \quad \text{with „critical ratio“ } \beta = \frac{c_u}{c_o + c_u}.$$
Back to **data example** of blood bags:

Looking at sample ACF and PACF,

$$\hat{\rho}_D(k)$$  $$\hat{\rho}_D;\text{part}(k)$$

significant autocorrelation visible, AR(1)-like.

**Question:** Possible to improve ordering strategy by utilizing serial dependence between demand counts?

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Modeling
Correlated Discrete Demand

INAR(1) Model
**Blood bags data example:** AR(1)-like autocorrelation.

⇒ **1st idea:** Apply basic Gaussian AR(1) model,

\[
D_t = \alpha \cdot D_{t-1} + \epsilon_t, \quad |\alpha| \leq 1.
\]

But does not preserve discreteness of demand counts.

⇒ **2nd idea:** Apply **INAR(1) model** (McKenzie, 1985),

\[
D_t = \alpha \circ D_{t-1} + \epsilon_t,
\]

where ‘\( \circ \)’ is **binomial thinning** operator, i.e.,

\[\alpha \circ D \sim \text{Bin}(D, \alpha)\]  (Steutel & van Harn, 1979).
Basics about INAR(1) processes:

- Intuitive interpretation:
  \[ D_t = \alpha \circ D_{t-1} + \epsilon_t \]
  Population at time \( t \) Survivors of time \( t-1 \) Immigration

- Homogeneous Markov chain, stationary solution satisfies
  \[
  \mu_D = \frac{\mu_\epsilon}{1 - \alpha}, \quad \sigma^2_D = \frac{\sigma^2_\epsilon}{\mu_\epsilon} + \alpha \frac{1}{1 + \alpha}.
  \]

- Autocorrelation function \( \rho_D(k) = \alpha^k \), i.e., AR(1)-type.

- One-step ahead transition probabilities:
  \[
  P(D_{t+1} = k | D_t = l) = \sum_{j=0}^{\min\{k,l\}} \binom{l}{j} \alpha^j (1 - \alpha)^{l-j} \cdot P(\epsilon_t = k - j).
  \]
Features of marginal distribution of observed demands $D_t$ determined through corresponding properties of innovations’ distribution, e.g.,

- $\epsilon_t \sim \text{Poi}(\lambda) \Rightarrow D_t \sim \text{Poi}(\frac{\lambda}{1-\alpha})$, Poisson INAR(1) model, equidispersion;
- $\epsilon_t \sim \text{NB}(n, \pi)$ (Schweer & Weiβ, 2014), NB INAR(1) model, overdispersion;
- $\epsilon_t \sim \text{ZIP}(\lambda, \omega)$ (Jazi et al., 2012), ZIP INAR(1) model, zero inflation.

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INAR(1) Model for Correlated Discrete Demand

Blood bags data example:

Estimated mean and variance are $\bar{d} \approx 2.792$ and $s_D^2/\bar{d} \approx 2.44$, i.e., strong degree of overdispersion. ACF $\hat{\rho}_D(1) \approx 0.266$.

<table>
<thead>
<tr>
<th>Model</th>
<th>ML estimates of Par. 1</th>
<th>Par. 2</th>
<th>Par. 3</th>
<th>AIC</th>
<th>BIC</th>
<th>Fitted model: $\mu_D$</th>
<th>$\sigma_D^2/\mu_D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>iid Poi $(\lambda)$</td>
<td>2.792</td>
<td></td>
<td></td>
<td>1157.9</td>
<td>1161.3</td>
<td>2.792</td>
<td>1</td>
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<tr>
<td>Poi INAR(1) $(\lambda, \alpha)$</td>
<td>2.322</td>
<td>0.167</td>
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<td>1135.8</td>
<td>1142.8</td>
<td>2.789</td>
<td>1</td>
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<tr>
<td>iid NB $(n, \pi)$</td>
<td>1.831</td>
<td>0.396</td>
<td></td>
<td>1039.2</td>
<td>1046.2</td>
<td>2.792</td>
<td>2.53</td>
</tr>
<tr>
<td>NB INAR(1) $(n, \pi, \alpha)$</td>
<td>1.370</td>
<td>0.374</td>
<td>0.179</td>
<td>1029.1</td>
<td>1039.6</td>
<td>2.789</td>
<td>2.67</td>
</tr>
</tbody>
</table>

⇒ NB INAR(1) model best choice for blood bags data.

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INAR(1) Implementation of Newsvendor Problem

Conditional approach
Idea behind INAR(1) implementation:
Utilize serial dependence by determining order quantity $Q$ for period $t + 1$ based on previous demand $D_t$, i.e., $Q_{t+1} = Q(D_t)$ varies with time $t$.

Expected cost $E[C(Q(D_t), D_{t+1})]$ depend on joint distribution of $(D_t, D_{t+1})$.

Monotonicity of expectation op. plus law of total expectation: expected cost $E[C(Q(D_t), D_{t+1})]$ minimized if $Q^*_t$ minimizes conditional expected cost $E[C(Q(D_t), D_{t+1}) \mid D_t]$. 

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INAR(1) Implementation of Newsvendor Problem

Remember: marginal approach

\[ Q^* = \min \left\{ Q \in \mathbb{N}_0 \mid F(Q) \geq \beta \right\} \quad \text{with "critical ratio"} \quad \beta = \frac{c_u}{c_o + c_u}. \]

Conditional approach:

Let \( F_{D_{t+1}|D_t=l}(k) \) be conditional CDF of INAR(1) process, define order quantity

\[ Q^*_{t+1}(D_t = l) = \min \left\{ Q \in \mathbb{N}_0 \mid F_{D_{t+1}|D_t=l}(Q) \geq \beta \right\}. \]
Expected cost against $\alpha$, cost model $80-20 \ (\beta = 0.80)$, Poi INAR(1) model with $\mu_D = 3$
(and corr. normal approximation):
Expected cost against $\beta$,
cost model $c_u = \beta \cdot $100 and $c_o = (1 - \beta) \cdot $100,

Poi INAR(1) model with $\mu_D = 1.5$ and different autocorr. levels:
Expected cost against $\alpha$, cost model $80–20$ ($\beta = 0.8$), NB INAR(1) model with $\mu_D = 3$ and $\sigma^2_D = 6$
(and corr. Poisson- resp. normal approximation):
**Expected cost** against $\alpha$, cost model $80–$20 ($\beta = 0.80$), NB INAR(1) model with $\mu_D = 3$ and different dispersion ratios:

\[
\begin{align*}
\text{NB model, } \mu_D &= 3.0 \\
\sigma_D^2 &= 4.5 \\
\sigma_D^2 &= 6.0 \\
\sigma_D^2 &= 9.0
\end{align*}
\]
Blood bags data: \( (\bar{d} \approx 2.79, \ s^2_D \approx 6.81, \ \hat{\rho}_D(1) \approx 0.27) \)

Fitted NB INAR(1) model, cost model $80–$20, gives

- marginal: \( Q^* = 5 \) and total cost $20 040;
- conditional: total cost $19 380.
Conclusions

- Novel implementation of newsvendor problem for autocorrelated discrete demands stemming from INAR(1) process.

- Comparison of dynamic INAR(1) implementation against traditional static implementation (also against approximating dynamic implementations).

- Dynamic INAR(1) implementation for appropriately specified conditional model always gives cost reduction, also much better than Gaussian AR(1) implementation.

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Thank You
for Your Interest!

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