Serial Dependence in Categorical Time Series

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Categorical Time Series and Categorical Processes

Basic concepts

Categorical process:

$(X_t)_{\mathbb{N}}$ with $\mathbb{N} = \{1, 2, \ldots\}$, where each

X_t takes one of **finite** number of **unordered** categories.

Categorical time series:

Realizations $(x_t)_{t=1,\ldots,T}$ from $(X_t)_{\mathbb{N}}$.

To simplify notations:

Range of $(X_t)_{\mathbb{N}}$ is coded as $\mathcal{V} = \{0, 1, \ldots, m\}$,

i. e., $P(X_t = 0) = 1 - \sum_{j=1}^{m} P(X_t = j)$.

Examples:

- Network traffic data;
- medical diagnoses of an examiner;
- text analysis, part-of-speech tagging;
- genetic or protein sequences;
- quality of produced items; . . .

Notations for time-invariant probabilities:

If $(X_t)_{\mathbb{N}}$ (strictly) stationary, then:

- marginal probabilities $p_i := P(X_t = i) \in (0;1)$.
  $\boldsymbol{p} := (p_0, \ldots, p_m)^\top$, and
  $s_k(\boldsymbol{p}) := \sum_j p_j^k$ for $k \in \mathbb{N}$; obviously $s_1(\boldsymbol{p}) = 1$.
- bivariate probabilities $p_{ij}(k) := P(X_t = i, X_{t-k} = j)$,
  conditional probabilities $p_{i|j}(k) := P(X_t = i \mid X_{t-k} = j)$.

Models for stationary categorical processes

with range $\mathcal{V} = \{0, \ldots, m\}$:

- **p$^{\text{th}}$ order Markov model**: $(m+1)^{\mathrm{p}} \cdot m$ parameters;
- **variable length M. m.** of Bühlmann & Wyner (1999): more parsimonious, but model choice difficult;
- **MTD(p) model** of Raftery (1985): still $m(m+1) + \mathrm{p} - 1$ parameters;
- **NDARMA(p, q) models** of Jacobs & Lewis (1983): $m + \mathrm{p} + \mathrm{q}$ parameters, also non-Markovian dependence.

$(X_t)_{\mathbb{Z}}$, $(\epsilon_t)_{\mathbb{Z}}$: categorical processes with range $\mathcal{V} = \{0, \ldots, m\}$;

$(\epsilon_t)_{\mathbb{Z}}$: i.i.d. with marginal $\boldsymbol{p}$, ϵ_t independent of $(X_s)_{s<t}$.

For $\varphi_{\mathsf{q}} > 0$, with $\phi_{\mathsf{p}} > 0$ if $\mathsf{p} \geq 1$, let

$$\boldsymbol{D}_t = (\alpha_{t,1}, \ldots, \alpha_{t,\mathsf{p}}, \beta_{t,0}, \ldots, \beta_{t,\mathsf{q}}) \sim MULT(1; \phi_1, \ldots, \phi_{\mathsf{p}}, \varphi_0, \ldots, \varphi_{\mathsf{q}})$$

be i.i.d. and independent of $(\epsilon_t)_{\mathbb{Z}}$, $(X_s)_{s<t}$.

$(X_t)_{\mathbb{Z}}$ is **NDARMA(p, q) process** if

$$X_t \;=\; \alpha_{t,1} \cdot X_{t-1} + \ldots + \alpha_{t,\mathsf{p}} \cdot X_{t-\mathsf{p}} \;+\; \beta_{t,0} \cdot \epsilon_t + \ldots + \beta_{t,\mathsf{q}} \cdot \epsilon_{t-\mathsf{q}}.$$

Some properties of NDARMA processes:

- marginal distribution $P(X_t = j) = p_j$;
- concerning the four measures of serial dependence discussed below, $\kappa(k)$, $\kappa^*(k)$, $\upsilon(k)$, $\tau(k)$, we have equality,

$$\kappa(k) \;=\; \kappa^*(k) \;=\; \upsilon(k) \;=\; \tau(k),$$

  and these measures satisfy **Yule-Walker-type equations**, i. e., **ARMA-like serial dependence structure**;

  (Weiß & Göb, 2008)

- ψ-mixing with exponent. decreasing weights (Weiß, 2013).

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Categorical Time Series Analysis

Basic concepts

Let $(X_t)_{\mathbb{N}}$ be stationary.

Measures of location:

only **mode** of X_t in use, i. e.,

value $i \in \mathcal{V}$ such that $p_i \geq p_j$ for all $j \in \mathcal{V}$.

Often not uniquely determined (e. g., uniform distribution).

Measures of dispersion:

dispersion $\approx$ quantity of uncertainty, two extremes:

maximal dispersion if all p_j equal (**uniform distribution**),

minimal disp. if $p_j = 1$ for one $j \in \mathcal{V}$ (**one-point distrib.**).

Common standardized measures of dispersion:

Gini index: $\nu_{\mathrm{G}}(X) := \frac{m+1}{m}\left(1-\sum_j p_j^2\right).$

Entropy: $\nu_{\mathrm{E}}(X) := -\frac{1}{\ln(m+1)}\sum_j p_j \ln p_j.$

Properties:

- continuous and symmetric function of $p_0,\ldots,p_m$,
- range $[0;1]$,

  1 iff uniform distribution, 0 iff one-point distribution.

Alternative measures of dispersion: Weiß & Göb (2008).

Weiß & Göb (2008): **(signed) serial dependence**.

Stationary categorical process $(X_t)_{\mathbb{N}}$ said to be

- **serially independent** at lag $k \in \mathbb{N}$

  if $p_{i|j}(k) = p_i$ (i. e., $p_{ij}(k) = p_i p_j$) for any $i, j \in \mathcal{V}$;

- perfectly **serially dependent** at lag $k \in \mathbb{N}$

  if for any $j \in \mathcal{V}$,

  conditional distribution $p_{i|j}(k)$ is one-point distribution.

(. . .)

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In case of perfect serial dependence at lag $k \in \mathbb{N}$:

- perfect **positive dependence**

  if $p_{i|j}(k) = 1$ iff $i = j$ for all $i, j \in \mathcal{V}$;

- perfect **negative dependence** if all $p_{i|i}(k) = 0$.

Measures of unsigned dependence:

(Weiß, 2011; Weiß & Göb, 2008)

Goodman and Kruskal's τ:

$$\tau(k) \;=\; \sqrt{\sum_{i,j} \frac{\big(p_{ij}(k) - p_i p_j\big)^2}{p_j\big(1 - s_2(\boldsymbol{p})\big)}} \qquad \text{with range } [0;1],$$

Cramer's v:

$$v(k) \;=\; \sqrt{\frac{1}{m} \cdot \sum_{i,j} \frac{\big(p_{ij}(k) - p_i p_j\big)^2}{p_i p_j}} \qquad \text{with range } [0;1].$$

Some properties:

- $X_t,\ X_{t-k}$ independent $\quad\Leftrightarrow\quad \tau(k) = v(k) = 0.$
- X_t depends perfectly on $X_{t-k} \quad\Leftrightarrow\quad \tau(k) = v(k) = 1.$

Further measures of unsigned dependence, e. g.,

- (auto-)mutual information function (Dehnert et al., 2003; Biswas & Guha, 2009);
- Pearson measure (Weiß & Göb, 2008, Bagnato et al., 2012);
- Goodman and Kruskal's λ, uncertainty coefficient, Φ^2-measure, Sakoda measure (Weiß & Göb, 2008);
- auto-odds-ratio function (Biswas & Song, 2009); ...

Measures of signed dependence:

(Weiß, 2011; Weiß & Göb, 2008)

Cohen's κ:

$$\kappa(k) \;=\; 1 - \frac{1 - \sum_j p_{jj}(k)}{1 - s_2(\boldsymbol{p})} \qquad \text{with range } [-\tfrac{s_2(\boldsymbol{p})}{1-s_2(\boldsymbol{p})} \;;\; 1],$$

Modified κ:

$$\kappa^*(k) \;=\; \tfrac{1}{m} \cdot \big(\textstyle\sum_j \;\; p_{j|j}(k) \;\; - 1 \big) \qquad \text{with range } [-\tfrac{1}{m}; 1].$$

Some properties:

- $X_t,\, X_{t-k}$ independent $\quad\Rightarrow\quad \kappa(k) = \kappa^*(k) = 0.$
- $X_t,\, X_{t-k}$ perf. positively dep. $\quad\Leftrightarrow\quad \kappa(k) = \kappa^*(k) = 1.$
- $X_t,\, X_{t-k}$ perf. negatively dep. $\quad\Rightarrow\quad \kappa(k), \kappa^*(k)$ minimal.

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Empirical Measures of Serial Dependence

Asymptotic Properties

Empirical Measures of Serial Dependence

Let $(X_t)_{\mathbb{N}}$ be stationary,

we have segment $X_1, \ldots, X_T$ of $(X_t)_{\mathbb{N}}$.

$N_i(T)$ number of variables $X_t = i$ in segment,

$N_{ij}(k,T)$ number of pairs $(X_t, X_{t-k}) = (i,j)$ in segment.

Simple unbiased estimators for p_i and $p_{ij}(k)$:

$$\widehat{p}_i(T) := \tfrac{1}{T} \cdot N_i(T) \qquad \text{and} \qquad \widehat{p}_{ij}(k,T) := \tfrac{1}{T-k} \cdot N_{ij}(k,T).$$

Empirical Measures of Dispersion

Gini index: $\nu_{\mathrm{G}}(X_t) \;=\; \frac{m+1}{m} \cdot \big(1 - s_2(\boldsymbol{p})\big).$

Empirical Gini index:

$$\widehat{\nu}_{\mathrm{G}} \;:=\; \frac{m+1}{m} \cdot \frac{T}{T-1} \cdot \big(1 - s_2(\widehat{\boldsymbol{p}}(T))\big),$$

exactly unbiased if $X_1, \ldots, X_T$ be **i.i.d.** (Weiß, 2011).

Theorem: (Weiß, 2013)

Let $X_1, \ldots, X_T$ stem from stationary NDARMA(p, q) process,

let $c \;:=\; 1 + 2 \cdot \sum_{k=1}^{\infty} \kappa(k).$

Then $1 - s_2\big(\widehat{\boldsymbol{p}}(T)\big)$ asymptotically normal

with mean $1 - s_2(\boldsymbol{p})$ and variance $\frac{4c}{T} \cdot \big(s_3(\boldsymbol{p}) - s_2^2(\boldsymbol{p})\big)$.

Empirical Measures of Dispersion

Entropy: $\nu_{\mathrm{E}}(X_t) \;=\; -\frac{1}{\ln(m+1)} \sum_{j=0}^{m} p_j \ln p_j.$

Empirical entropy:

$$\widehat{\nu}_{\mathrm{E}}(X) \;:=\; -\frac{1}{\ln(m+1)} \sum_{j=0}^{m} \widehat{p}_j(T) \ln \widehat{p}_j(T).$$

Theorem: (Weiß, 2013)

Let $X_1, \ldots, X_T$ stem from stationary NDARMA(p, q) process,

let $c \;:=\; 1 + 2 \cdot \sum_{k=1}^{\infty} \kappa(k).$

Then $-\sum_{j=0}^{m} \widehat{p}_j(T) \ln \widehat{p}_j(T)$ asymptotically normal

with mean $-\sum_{j=0}^{m} p_j \ln p_j$

and variance $\frac{c}{T} \cdot \Big(\sum_{i=0}^{m} p_i \cdot (\ln p_i)^2 \;-\; \big(\sum_{i=0}^{m} p_i \cdot \ln p_i \big)^2 \Big).$

Empirical Measures of Serial Dependence

Cohen's κ: $$\kappa(k) \;=\; 1 - \frac{1 - \sum_j p_{jj}(k)}{1 - s_2(\boldsymbol{p})}.$$

Empirical Cohen's κ:

$$\widehat{\kappa}(k) \;:=\; 1 + \tfrac{1}{T} \;-\; \frac{1 - \sum_j \widehat{p}_{jj}(k,T)}{1 - \sum_j \widehat{p}_j(T)^2}.$$

Theorem: (Weiß, 2011)

If $X_1, \ldots, X_T$ are **i.i.d.**, then

$\widehat{\kappa}(k)$ asymptotically normally distributed with

$$E\big[\widehat{\kappa}(k)\big] \;=\; 0 \;+\; O(T^{-2}),$$

$$V\big[\widehat{\kappa}(k)\big] \;=\; \tfrac{1}{T} \cdot \big(1 \;-\; \tfrac{1+2s_3(\boldsymbol{p})-3s_2(\boldsymbol{p})}{(1-s_2(\boldsymbol{p}))^2}\big) \;+\; O(T^{-2}).$$

Genome of Bovine Leukemia Virus: (Weiß & Göb, 2008)

Range of size 4 ($\Rightarrow m = 3$),

coding a $\mapsto$ 0, c $\mapsto$ 1, g $\mapsto$ 2, t $\mapsto$ 3,

length $T = 8419$.

Estimated marginal probabilities:

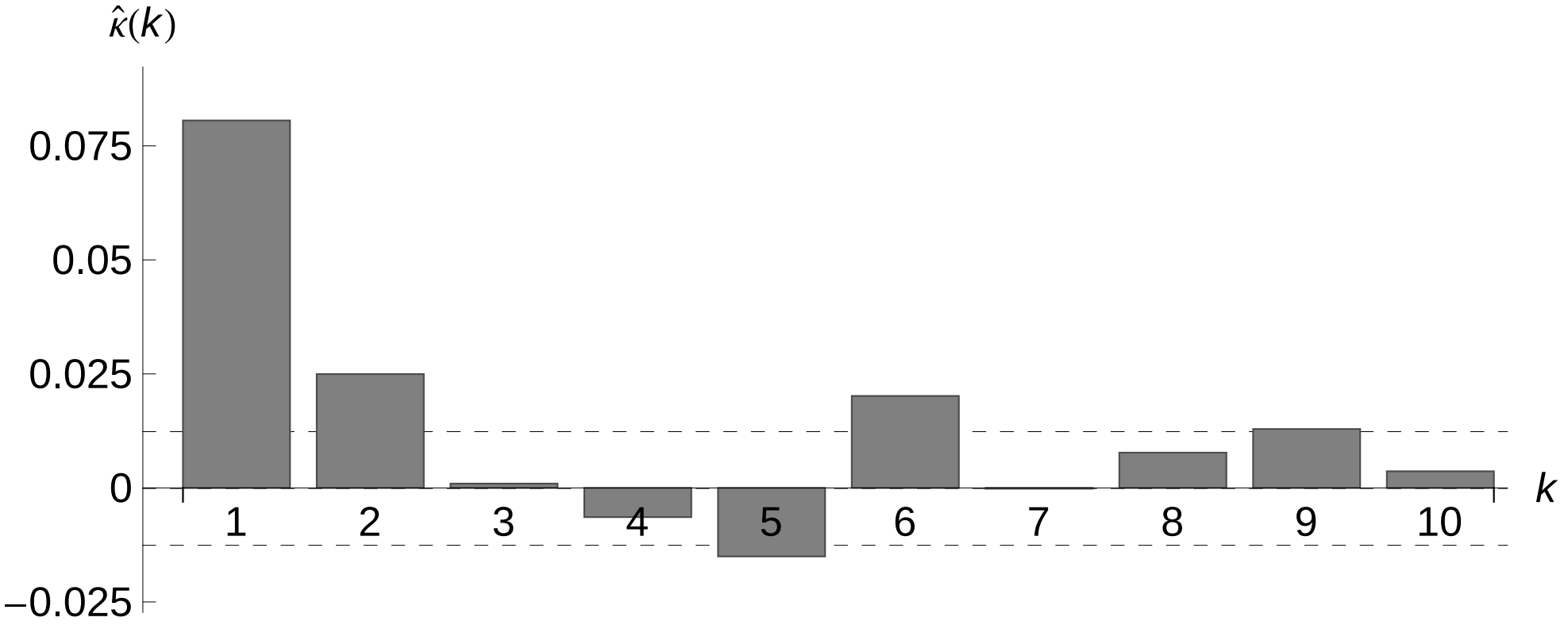
$\hat{p}_0 = 0.220$, $\hat{p}_1 = 0.331$, $\hat{p}_2 = 0.210$, $\hat{p}_3 = 0.239$

$\Rightarrow$ Gini index $\hat{\nu}_{\mathrm{G}} \approx 0.988$ (s.e. 0.0016),

entropy $\hat{\nu}_{\mathrm{E}} \approx 0.987$ (s.e. 0.0016) (strong dispersion).

Genome of Bovine Leukemia Virus: (continued)

(Approximate) asymptotic standard error for $\widehat{\kappa}(k)$: 0.00636.



Theorem 5.1.1 in Weiß (2013):

Asymptotics of $\widehat{\kappa}(k)$ for stationary NDARMA(p, q) process.

Corollary:

Let $X_1, \ldots, X_T$ stem from stationary DAR(1) process,

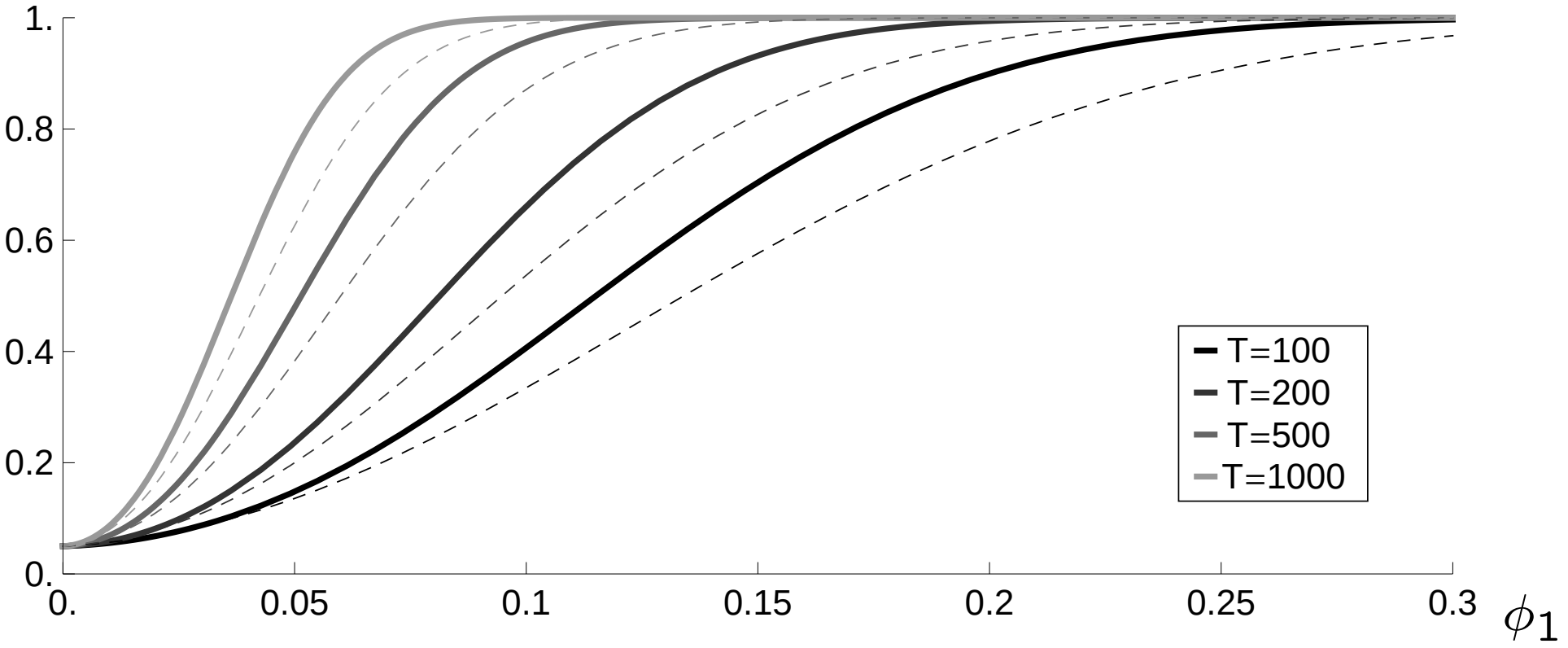
then $\widehat{\kappa}(1)$ asymptotically normal with mean ϕ_1 and variance

$$\frac{1}{T}\Big(2\cdot(2\phi_1-1)\cdot(1-\phi_1)\cdot\frac{s_3(\boldsymbol{p})-s_2^2(\boldsymbol{p})}{(1-s_2(\boldsymbol{p}))^2} + \frac{(1-\phi_1)\cdot s_2(\boldsymbol{p})}{1-s_2(\boldsymbol{p})} + \phi_1(1-\phi_1)\Big).$$

Applications: parameter estimation, power analysis ...

Power functions if testing $\widehat{\kappa}(1)$ against 0:

(marginals $(0.2, 0.2, 0.25, 0.35)^\top$ (solid) and $(0.05, 0.1, 0.15, 0.7)^\top$ (dashed))



Analogous results also available for **empirical modified** κ,

$$\widehat{\kappa}^*(k) \;:=\; \frac{1}{m}\cdot\Big(\sum_j \frac{\widehat{p}_{jj}(k,T)}{\widehat{p}_j(T)} - 1\Big) + \frac{1}{T},$$

see Weiß (2011,2013).

But:

Practical issues like division by zero, finite-sample bias, ...

$\Rightarrow$ **empirical Cohen's** κ

preferable measure of *signed* serial dependence.

Empirical Measures of Serial Dependence

Cramer's v: $$v(k) = \sqrt{\frac{1}{m} \cdot \sum_{i,j=0}^{m} \frac{\big(p_{ij}(k) - p_i p_j\big)^2}{p_i p_j}}.$$

Empirical Cramer's v:

$$\widehat{v}^2(k) = \frac{1}{m} \cdot \sum_{i,j=0}^{m} \frac{\big(\widehat{p}_{ij}(k,T) - \widehat{p}_i(T)\widehat{p}_j(T)\big)^2}{\widehat{p}_i(T)\widehat{p}_j(T)}.$$

Theorem: (Weiß, 2013)

If $X_1, \ldots, X_T$ are **i.i.d.**, then

$\widehat{v}(k)$ asymptotically distributed via

$$T \cdot m \cdot \widehat{v}^2(k) \underset{\text{a.}}{\sim} \chi^2_{m^2}.$$

Asymptotic χ^2-distribution (confirmed through simulations) of empirical Cramer's v implies non-zero mean although $v(k) = 0$ for i.i.d. data!

$\Rightarrow \widehat{v}(k)$ problematic as estimator of $v(k)$;
similar problems with any unsigned measure.

Certainly useful for uncovering significant dependence.

Simulations: Less powerful than Cohen's κ.

Bovine data: $\widehat{v}(k)$ as 0.1134, 0.0445, 0.0281, 0.0222, ...
Crit. value 0.0259 (level 0.05) $\Rightarrow \widehat{v}(1)$, $\widehat{v}(2)$, $\widehat{v}(3)$ significant.

Analogous results also available for

empirical Goodman and Kruskal's τ,

$$\widehat{\tau}^2(k) \;=\; \sum_{i,j=0}^{m} \frac{\big(\widehat{p}_{ij}(k,T) - \widehat{p}_i(T)\widehat{p}_j(T)\big)^2}{\widehat{p}_j(T)\big(1 - s_2(\widehat{\boldsymbol{p}}(T))\big)},$$

see Weiß (2011,2013).

But asymptotic distribution now

quadratic form distribution instead of simple χ^2-distribution

$\Rightarrow$ $\widehat{\tau}(k)$ less attractive from practical point of view.

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Conclusions

. . . and Future Research

Conclusions

- Both unsigned and signed empirical measures of serial dependence in categorical time series available, readily applicable to uncover significant dependence.
- The signed measure Cohen's κ appears most attractive, because also useful for parameter estimation, and good agreem. betw. asymptotic & finite-sample properties.
- Gini index useful measure of dispersion in categ. time series.

Future research:

- Effect of estimated parameters;
- unique framework for serial dependence.

Thank You

for Your Interest!

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