Empirical Measures of Signed Serial Dependence in Categorical Time Series

TECHNISCHE UNIVERSITÄT DARMSTADT

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All references mentioned in this talk correspond to the references in this article.

Categorical Time Series Analysis

Brief Review

Categorical process:

$(X_t)_{\mathbb{N}}$ with $\mathbb{N} = \{1, 2, \ldots\}$, where each

X_t takes one of **finite** number of **unordered** categories.

Categorical time series:

Realizations $(x_t)_{t=1,\ldots,T}$ from $(X_t)_{\mathbb{N}}$.

To simplify notations:

Range of $(X_t)_{\mathbb{N}}$ is coded as $\mathcal{V} = \{0, 1, \ldots, m\}$,

i. e., $P(X_t = 0) = 1 - \sum_{j=1}^{m} P(X_t = j)$.

Notations for time-invariant probabilities:

If $(X_t)_{\mathbb{N}}$ (strictly) stationary, then:

- marginal probabilities $p_i := P(X_t = i) \in (0;1)$.
  $\boldsymbol{p} := (p_0, \ldots, p_m)^\top$, and
  $s_k(\boldsymbol{p}) := \sum_j p_j^k$ for $k \in \mathbb{N}$; obviously $s_1(\boldsymbol{p}) = 1$.
- bivariate probabilities $p_{ij}(k) := P(X_t = i, X_{t-k} = j)$,
  conditional probabilities $p_{i|j}(k) := P(X_t = i \mid X_{t-k} = j)$.

Let $(X_t)_{\mathbb{N}}$ be stationary.

Measures of location:

only **mode** of X_t in use, i. e.,

value $i \in \mathcal{V}$ such that $p_i \geq p_j$ for all $j \in \mathcal{V}$.

Often not uniquely determined (e. g., uniform distribution).

Measures of dispersion:

dispersion $\approx$ quantity of uncertainty, two extremes:

maximal dispersion if all p_j equal (**uniform distribution**),

minimal disp. if $p_j = 1$ for one $j \in \mathcal{V}$ (**one-point distrib.**).

Most simple measure of dispersion: **Gini index** of X_t,

$$\nu_{\mathsf{G}}(X_t) \; := \; \tfrac{m+1}{m}\cdot\Big(1-\textstyle\sum_j\; p_j^2\Big) \;=\; \tfrac{m+1}{m}\cdot\Big(1-s_2(\boldsymbol{p})\Big).$$

- continuous and symmetric function of $p_1,\ldots,p_{m+1}$,
- range $[0;1]$,
- maximal value 1 iff uniform distribution,
- minimal value 0 iff one-point distribution.

Alternative measures of dispersion: Weiß & Göb (2008).

Weiß & Göb (2008): **signed serial dependence**.

Stationary categorical process $(X_t)_{\mathbb{N}}$ said to be

- **serially independent** at lag $k \in \mathbb{N}$

  if $p_{i|j}(k) = p_i$ (i. e., $p_{ij}(k) = p_i p_j$) for any $i, j \in \mathcal{V}$;

- perfectly **serially dependent** at lag $k \in \mathbb{N}$

  if for any $j \in \mathcal{V}$,

  conditional distribution $p_{i|j}(k)$ is one-point distribution.

(. . .)

(. . .)

In case of perfect serial dependence at lag $k \in \mathbb{N}$:

- perfect **positive dependence**

  if $p_{i|j}(k) = 1$ iff $i = j$ for all $i, j \in \mathcal{V}$;

- perfect **negative dependence** if all $p_{i|i}(k) = 0$.

Measures of signed dependence:

(Weiß, 2011; Weiß & Göb, 2008)

Cohen's κ:

$$\kappa(k) \;=\; 1-\frac{1-\sum_j p_{jj}(k)}{1-s_2(\boldsymbol{p})} \qquad \text{with range } \left[-\tfrac{s_2(\boldsymbol{p})}{1-s_2(\boldsymbol{p})}\ ;\ 1\right],$$

Modified κ:

$$\kappa^*(k) \;=\; \tfrac{1}{m}\cdot\Big(\sum_j\ p_{j|j}(k)\ -1\Big) \qquad \text{with range } [-\tfrac{1}{m};1].$$

Some properties:

- X_t, X_{t-k} independent $\quad\Rightarrow\quad \kappa(k)=\kappa^*(k)=0$.
- X_t, X_{t-k} perf. positively dep. $\quad\Leftrightarrow\quad \kappa(k)=\kappa^*(k)=1$.
- X_t, X_{t-k} perf. negatively dep. $\quad\Rightarrow\quad \kappa(k),\kappa^*(k)$ minimal.

Empirical Measures of Signed Serial Dependence

Asymptotic Properties

Just to remember . . .

Cohen's κ:

$$\kappa(k) \;=\; 1-\frac{1-\sum_j p_{jj}(k)}{1-s_2(\boldsymbol{p})} \qquad \text{with range } \left[-\tfrac{s_2(\boldsymbol{p})}{1-s_2(\boldsymbol{p})}\;;\;1\right],$$

Modified κ:

$$\kappa^*(k) \;=\; \tfrac{1}{m}\cdot\Big(\sum_j\; p_{j|j}(k)\;-1\Big) \qquad \text{with range } [-\tfrac{1}{m};1],$$

Gini index ν_{G}:

$$\nu_{\mathrm{G}}(X_t) \;=\; \tfrac{m+1}{m}\cdot\Big(1-s_2(\boldsymbol{p})\Big) \qquad \text{with range } [0;1].$$

Let $(X_t)_{\mathbb{N}}$ be stationary,

we have segment $X_1, \ldots, X_T$ of $(X_t)_{\mathbb{N}}$.

$$N_i(T) \quad \text{number of variables } X_t = i \text{ in segment,}$$
$$N_{ij}(k,T) \quad \text{number of pairs } (X_t, X_{t-k}) = (i,j) \text{ in segment.}$$

Simple unbiased estimators for p_i and $p_{ij}(k)$:

$$\widehat{p}_i(T) \;:=\; \tfrac{1}{T} \cdot N_i(T) \qquad \text{and} \qquad \widehat{p}_{ij}(k,T) \;:=\; \tfrac{1}{T-k} \cdot N_{ij}(k,T).$$

Lemma:

Let $X_1, \ldots, X_T$ be **i.i.d.**

Estimator $1 - \sum_j \widehat{p}_j(T)^2$ of $1 - s_2(\boldsymbol{p})$ satisfies

$$E\Big[1 - \textstyle\sum_j \widehat{p}_j(T)^2\Big] \;=\; 1 - s_2(\boldsymbol{p}) \;-\; \tfrac{1}{T} \cdot \Big(1 - s_2(\boldsymbol{p})\Big),$$

$$V\Big[1 - \textstyle\sum_j \widehat{p}_j(T)^2\Big] \;=\; \tfrac{4}{T} \cdot \Big(s_3(\boldsymbol{p}) - s_2^2(\boldsymbol{p})\Big) \;+\; O(T^{-2}).$$

$\Rightarrow$ Define exactly unbiased **empirical Gini index** via

$$\widehat{\nu}_{\mathrm{G}} \;:=\; \tfrac{m+1}{m} \cdot \tfrac{T}{T-1} \cdot \Big(1 - \textstyle\sum_j \widehat{p}_j(T)^2\Big).$$

Lemma:

Let $X_1, \ldots, X_T$ be **i.i.d.**

Enumerator $1 - \sum_i p_{ii}(k)$ of Cohen's κ:

$$E\Big[1 - \textstyle\sum_i\ \widehat{p}_{ii}(k,T)\Big]\ =\ 1 - s_2(\boldsymbol{p}),$$

$$V\Big[1 - \textstyle\sum_i\ \widehat{p}_{ii}(k,T)\Big]\ =$$
$$\tfrac{1}{T-k}\cdot\Big(s_2(\boldsymbol{p})\big(1 - s_2(\boldsymbol{p})\big)\ +\ 2\big(s_3(\boldsymbol{p}) - s_2^2(\boldsymbol{p})\big)\Big)\ +\ O(T^{-2}).$$

Theorem: Define **empirical Cohen's** κ as

$$\widehat{\kappa}(k) \;:=\; 1+\tfrac{1}{T} \;-\; \frac{1-\sum_j \widehat{p}_{jj}(k,T)}{1-\sum_j \widehat{p}_j(T)^2}.$$

If $X_1,\ldots,X_T$ is **i.i.d.**, then

$\widehat{\kappa}(k)$ asymptotically normally distributed with

$$E\big[\widehat{\kappa}(k)\big] \;=\; 0 \;+\; O(T^{-2}),$$

$$V\big[\widehat{\kappa}(k)\big] \;=\; \tfrac{1}{T}\cdot\Big(1 \;-\; \tfrac{1+2s_3(\boldsymbol{p})-3s_2(\boldsymbol{p})}{(1-s_2(\boldsymbol{p}))^2}\Big) \;+\; O(T^{-2}).$$

Theorem: Define **empirical modified** κ as

$$\widehat{\kappa}^*(k) \;:=\; \frac{1}{m}\cdot\Big(\sum_j \frac{\widehat{p}_{jj}(k,T)}{\widehat{p}_j(T)} - 1\Big) \;+\; \frac{1}{T}.$$

If $X_1,\ldots,X_T$ is **i.i.d.**, then

$\widehat{\kappa}^*(k)$ asymptotically normally distributed with

$$E\big[\widehat{\kappa}^*(k)\big] \;=\; 0 \;+\; O(T^{-2}),$$

$$V\big[\widehat{\kappa}^*(k)\big] \;=\; \tfrac{1}{m\cdot T} \;+\; O(T^{-2}).$$

Empirical Measures of Signed Serial Dependence

An Application

Measured serial dependence at lag k called

significantly different from 0 if

$$\big|\hat{\kappa}(k)\big| \;>\; c\cdot\sqrt{\tfrac{1}{T}\cdot\Big(1\;-\;\tfrac{1+2s_3(\hat{\boldsymbol{p}})-3s_2(\hat{\boldsymbol{p}})}{(1-s_2(\hat{\boldsymbol{p}}))^2}\Big)},\qquad \text{or}$$

$$\big|\hat{\kappa}^*(k)\big| \;>\; c\cdot\sqrt{\tfrac{1}{m\cdot T}}.$$

Common choice: $c = 1.96$ ($\approx$ significance level 5 %).

Concerning $\hat{\kappa}(k)$, we used $\hat{\boldsymbol{p}} := \hat{\boldsymbol{p}}(T)$ instead of true $\boldsymbol{p}$, since latter hardly known in practice.

Data Example:

Genome of Bovine Leukemia Virus, as in Weiß & Göb (2008).

Range of size 4 ($\Rightarrow m = 3$),

coding a $\mapsto$ 0, c $\mapsto$ 1, g $\mapsto$ 2, t $\mapsto$ 3,

length $T = 8419$.

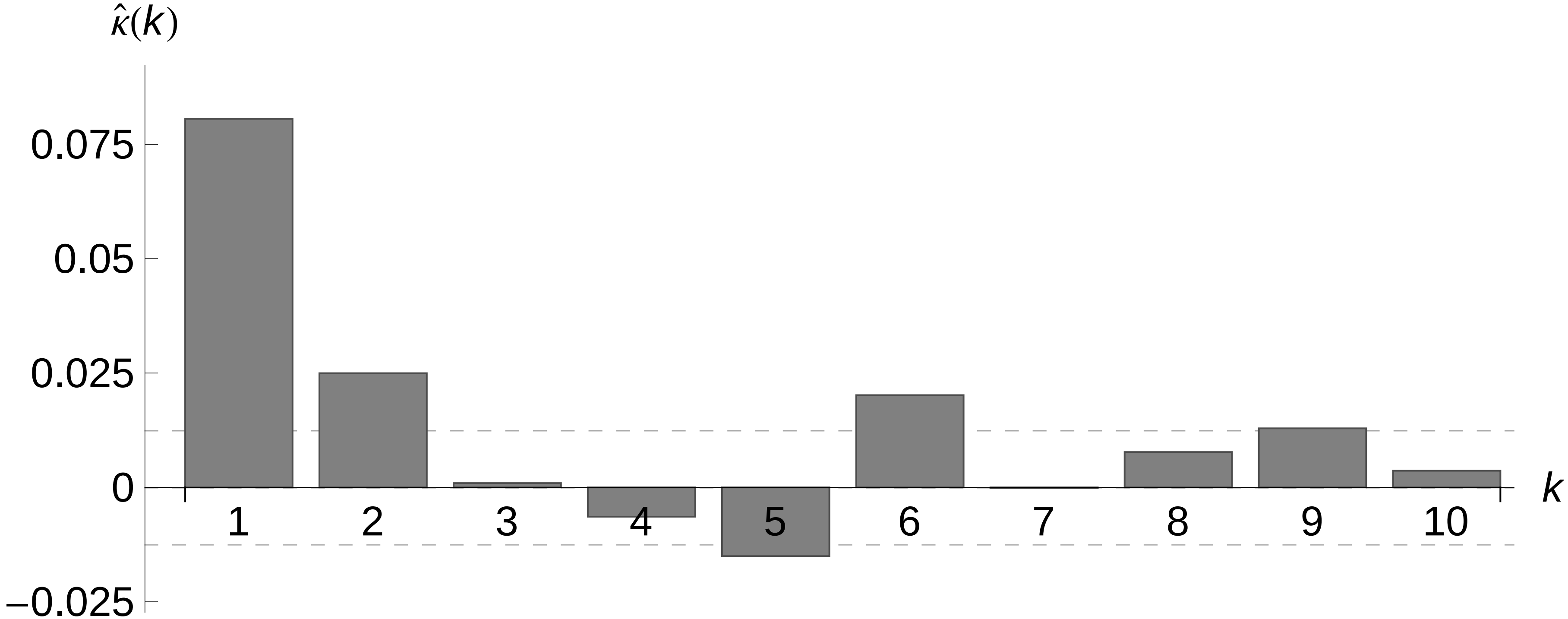
Estimated marginal probabilities:

$\widehat{p}_0 = 0.220$, $\widehat{p}_1 = 0.331$, $\widehat{p}_2 = 0.210$, $\widehat{p}_3 = 0.239$

$\Rightarrow$ Gini index $\widehat{\nu}_{\mathsf{G}} \approx 0.988$ (strong dispersion).

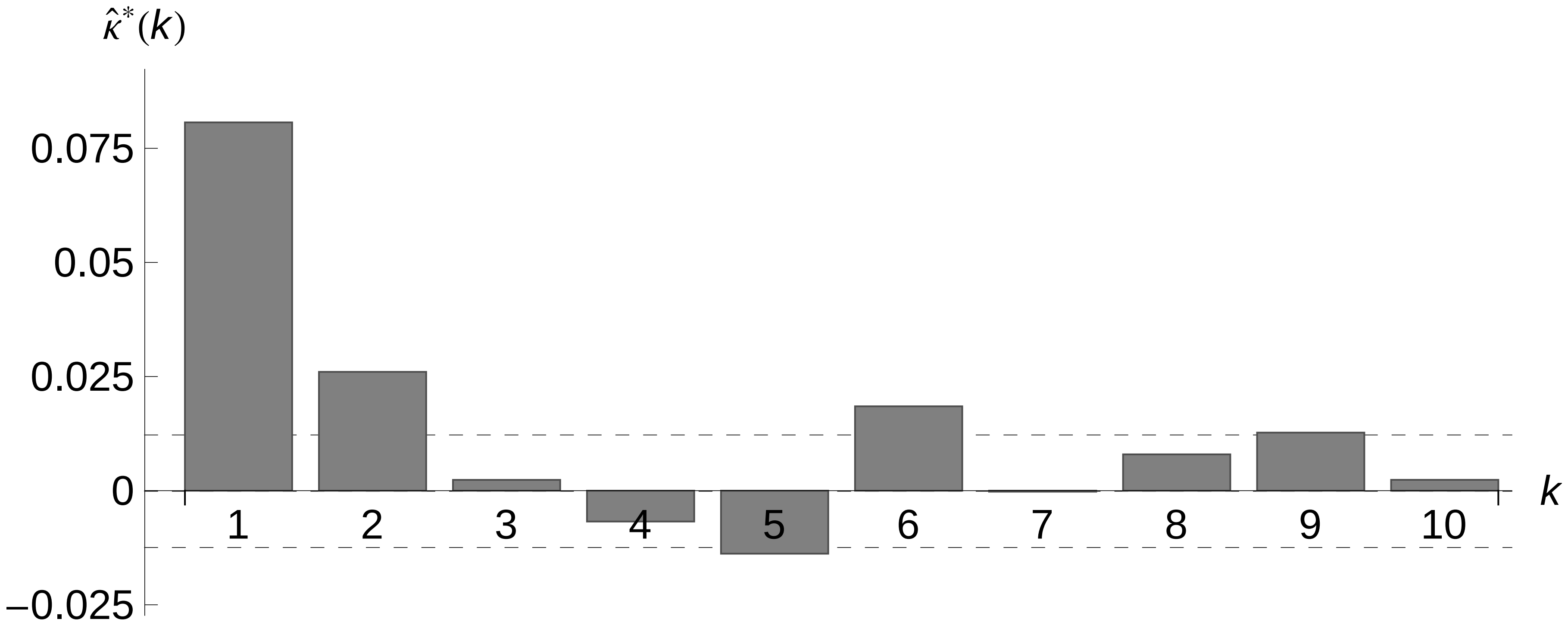
Data Example: (continued)

(Approximate) asymptotic standard error for $\widehat{\kappa}(k)$: 0.00636.



Data Example: (continued)

Asymptotic standard error for $\hat{\kappa}^*(k)$: 0.00629.



Empirical Measures of Signed Serial Dependence

Finite-Sample Properties

Selected results of simulation study,

detailed tables in Weiß (2011).

Models with range of size 4 (i. e., $m = 3$).

Study of true **significance level**:

'i.i.d.-1': $\boldsymbol{p}_1 = (0.20, 0.20, 0.25, 0.35)^\top$, $\nu_{\mathsf{G}}(X) = 0.98$

'i.i.d.-2': $\boldsymbol{p}_2 = (0.05, 0.10, 0.15, 0.70)^\top$, $\nu_{\mathsf{G}}(X) = 0.633$.

Design of simulation study (continued):

Study of true **power**:

'DAR(1)': $\boldsymbol{p}_1$ and $\phi = 0.25 \Rightarrow \kappa(k) = \kappa^*(k) = 0.25^k$.

'DMA(1)': $\boldsymbol{p}_1$ and $\varphi = 0.25 \Rightarrow$

$\kappa(1) = \kappa^*(1) = 0.1875$, $\kappa(k) = \kappa^*(k) = 0$ for $k \geq 2$.

'NegMarkov': $\boldsymbol{p}_1$ and $\alpha = 0.5 \Rightarrow$

$\kappa(1) = -0.2678$, $\kappa(2) = 0.0818$, $\kappa(3) = -0.0276$, ...

$\kappa^*(1) = -0.2500$, $\kappa^*(2) = 0.0719$, $\kappa^*(3) = -0.0232$, ...

Empirical rejection rates for $\widehat{\kappa}(k)$:

	i.i.d.-1				i.i.d.-2			
$k \setminus T$	100	200	500	1000	100	200	500	1000
1	5.1	4.7	4.8	4.9	5.0	4.6	4.7	4.9
2	4.9	4.9	4.8	5.5	5.5	4.9	4.9	5.4
3	5.1	4.9	5.0	5.1	5.2	5.2	5.4	5.0
4	5.1	5.0	5.0	5.2	5.6	5.6	5.2	4.7
5	5.2	5.1	5.0	5.1	5.7	5.2	5.1	5.1

$\Rightarrow$ always close to nominal level of 5 %.

Empirical rejection rates for $\widehat{\kappa}^*(k)$:

	i.i.d.-1				i.i.d.-2			
$k \setminus T$	100	200	500	1000	100	200	500	1000
1	5.1	4.6	4.8	4.9	3.7	4.4	4.5	5.2
2	4.8	4.9	5.0	5.5	3.9	4.2	4.6	5.0
3	4.8	5.0	5.0	4.9	3.8	4.0	4.7	5.0
4	5.1	4.9	5.0	5.2	3.9	4.3	5.1	4.4
5	5.2	4.9	4.9	5.0	3.5	4.0	4.6	5.0

$\Rightarrow$ for medium dispersion and $T \leq 200$,

even below nominal level of 5 %.

Empirical rejection rates for DAR(1) model:

	$\widehat{\kappa}(k)$				$\widehat{\kappa}^*(k)$			
$k \quad \backslash \quad T$	100	200	500	1000	100	200	500	1000
1	97.4	100.0	100.0	100.0	97.2	100.0	100.0	100.0
2	18.3	31.5	63.0	88.9	17.9	31.5	63.6	89.4
3	7.1	7.9	10.8	14.1	6.6	7.9	10.7	14.6
4	6.3	6.5	6.6	6.7	6.0	6.3	6.6	6.8
5	6.4	6.6	6.6	6.4	6.0	6.2	6.4	6.5

$\Rightarrow$ similar performance for both measures,

at least 1$^{\text{st}}$ order dependence nearly always detected.

Empirical rejection rates for DMA(1) model:

	$\widehat{\kappa}(k)$				$\widehat{\kappa}^*(k)$			
$k \setminus T$	100	200	500	1000	100	200	500	1000
1	86.7	99.3	100.0	100.0	87.4	99.3	100.0	100.0
2	5.4	5.7	5.6	5.7	5.3	5.5	5.6	5.6
3	5.9	5.8	5.9	5.6	5.7	5.5	5.9	5.5
4	5.9	5.8	5.7	5.4	5.5	5.7	6.0	5.6
5	6.3	6.1	5.6	5.7	5.9	6.0	5.7	5.8

$\Rightarrow$ similar performance for both measures.

For $T \geq 200$, 1$^{\text{st}}$ order dependence nearly always detected.

For $k \geq 2$, slightly larger than 5 %.

Empirical rejection rates for NegMarkov model:

	$\widehat{\kappa}(k)$				$\widehat{\kappa}^*(k)$			
$k \quad \backslash \quad T$	100	200	500	1000	100	200	500	1000
1	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
2	32.3	52.6	86.1	98.8	26.0	43.7	78.2	96.7
3	9.7	11.9	19.6	32.1	8.2	9.9	15.3	24.7
4	8.3	8.7	8.7	11.7	7.0	7.7	8.0	10.3
5	7.7	7.5	7.4	8.5	7.0	6.7	7.0	7.7

$\Rightarrow$ $\widehat{\kappa}^*(k)$ worse than $\widehat{\kappa}(k)$,

at least 1^{st} order dependence nearly always detected.

Conclusions

- Empirical measures of signed serial dependence, effective for identifying significant dependence.

- Finite-sample study shows that overally, $\widehat{\kappa}(k)$ is best choice. For $T \geq 500$, both measures perform equivalently.

- **Work in progress:** $\widehat{\kappa}(k)$, $\widehat{\kappa}^*(k)$ and also empirical measures of *unsigned* dependence for NDARMA processes. Empirical dispersion measures for NDARMA processes.

Thank You

for Your Interest!

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