Discover Patterns in Categorical Time Series using IFS.

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This talk is based on the paper

Weiß, C.H., Göb, R.:

Discover Patterns in Categorical Time Series using IFS.

Preprint 260, University of Würzburg, 2005.

Compare conference CD-ROM.

All references mentioned in this talk correspond to the references in this paper.

Iterated Function Systems (IFS)

Background

Iterated function systems (IFS) introduced by Barnsley (1988).

Definition: An *IFS* $\mathcal{F} = \{X;\ c_0, ..., c_{n-1}\}$ consists of

- metric space (X, d) and
- finite set of contractions $c_i : X \to X$, $i = 0, ..., n-1$.

Mapping $c : X \to X$ on metric space (X, d) is *contraction* if for all $x, y \in X$

$$d(c(x), c(y)) \ \leq \ \alpha d(x, y) \qquad \text{with } 0 < \alpha < 1.$$

Chaos algorithm for creating fractals (Barnsley (1988)):

Algorithm: INPUT: *IFS* $\mathcal{F} = \{X;\ c_0, ..., c_{n-1}\}$ and random sequence $i_1, i_2, ...$ with range $\{0, ..., n-1\}$.

1. Initialization: Choose arbitrary $\boldsymbol{x}_0 \in X$.
2. Iteration $t-1 \to t$: Calculate $\boldsymbol{x}_t = c_{i_t}(\boldsymbol{x}_{t-1})$.
3. Repeat step 2.

Iterated Function Systems (IFS)

Applications

Applications of IFS:

- Creating fractals, e.g., Sierpinski triangle, ferns, etc.,
- coding and compression of pictures, Jacquin (1992)
- 'action selection problem for autonomous agents' in robotics, Hoskins (1995)

Applications of IFS (continued):

- synthesis of textures, Chen & Chen (2003)
- analysis and visualization of nucleotide sequences, Jeffrey (1990), Wu et al. (1993)
- sequential pattern analysis. Weiß & Göb (2005)

Analysis of Categorical Time Series

Transformation via Chaos Algorithm

Given: Categorical Time Series $(a_{i_t})_{\mathcal{T}}$ over finite alphabet $\mathcal{A} = \{a_0, ..., a_{n-1}\}$.

Step 1: Transformation of alphabet $\mathcal{A}$ via mapping

$$\boldsymbol{b}: \ \mathcal{A} \to \mathbb{R}^p.$$

Result: Transformed alphabet $\mathcal{A}' = \{\boldsymbol{a}_0, ..., \boldsymbol{a}_{n-1}\} \subseteq \mathbb{R}^p$ with $\boldsymbol{a}_i = \boldsymbol{b}(a_i)$, and time series $(\boldsymbol{Z}_t)_{\mathcal{T}}$ with $\boldsymbol{Z}_t = \boldsymbol{b}(a_{i_t})$.

Step 2: Application of chaos algorithm.

IFS $\mathcal{F} = \{X;\ c_0, ..., c_{n-1}\}$ with linear contractions

$$c_i(\boldsymbol{x}) \;=\; \alpha \boldsymbol{x} \;+\; \beta \boldsymbol{a}_i \qquad \text{with } 0 < \alpha < 1 \text{ and } \beta > 0.$$

Result: Fractal series $\boldsymbol{X}_0 = \boldsymbol{x}_0, \boldsymbol{X}_1, \boldsymbol{X}_2, ...$ with

$$\boldsymbol{X}_t \;=\; \alpha \boldsymbol{X}_{t-1} \;+\; \beta \boldsymbol{Z}_t.$$

Explicitly: For $\boldsymbol{X}_0 := \boldsymbol{0}$

$$\boldsymbol{X}_t \;=\; \beta \cdot \sum_{j=0}^{t-1} \alpha^j \boldsymbol{Z}_{t-j}.$$

Examples:

- $\alpha = \frac{1}{2} = \beta$: *Chaos game representation*, Jeffrey (1990)
- $\alpha = \frac{1}{k} = \beta,\ k \in \mathbb{N}$: *W-curve*. Wu et al. (1993)

Analysis of Categorical Time Series

Main Results

Observation:

If alphabet transformation $\boldsymbol{b}$ and parameter α of IFS are chosen appropriately, then:

- If two segments $(\boldsymbol{Z}_i, ..., \boldsymbol{Z}_{i-k}) = (\boldsymbol{Z}_j, ..., \boldsymbol{Z}_{j-k})$, then points $\boldsymbol{X}_i$ and $\boldsymbol{X}_j$ will be *close* in $\mathbb{R}^p$,

- but if the segments differ, then points $\boldsymbol{X}_i$ and $\boldsymbol{X}_j$ will be *distant* in $\mathbb{R}^p$.

Idea:

1. Online-transformation of time series $\boldsymbol{Z}_1, \boldsymbol{Z}_2, \ldots$ into fractal series $\boldsymbol{X}_0, \boldsymbol{X}_1, \ldots$.

2. Screen the fractal series $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_T, \ldots$ for *close* elements (cluster).

   Points in a cluster correspond to occurrences of similar patterns in time series $\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_T$.

Possible realizations:

Transformation of original alphabet $\mathcal{A} = \{a_0, ..., a_{n-1}\}$

- into $\{-1, 1\}^p \subset \mathbb{R}^p$, i.e., edges of p-dim. unit cube
  $\rightarrow$ *cube transformation*,
  see section 5 in Weiß & Göb (2005).

- into unit circle S^1 in $\mathbb{R}^2$, via

$$\boldsymbol{b}(a_k) = (\cos(k \cdot \frac{2\pi}{n}), \sin(k \cdot \frac{2\pi}{n}))^T$$

  $\rightarrow$ *circle transformation*.

Circle Transformations

Background

Let $(\boldsymbol{Z}_t)_{t=1,2,\ldots}$ be series from alphabet $\mathcal{A}' \subset S^1$ resulting from *circle transformation*.

Let $(\boldsymbol{X}_t)_{t=0,1,\ldots}$ be corresponding fractal series

$$\boldsymbol{X}_t \;=\; \alpha \boldsymbol{X}_{t-1} \;+\; \beta \boldsymbol{Z}_t$$

with $0 < \alpha < 1$, $\beta > 0$ and $\boldsymbol{X}_0 = 0$.

For indices $i > j > k > 0$ we compare the segments

$$(\boldsymbol{Z}_i, \ldots, \boldsymbol{Z}_{i-k}) \text{ and } (\boldsymbol{Z}_j, \ldots, \boldsymbol{Z}_{j-k}).$$

Theorem 6.1: Let $d_n = \sqrt{2(1-\cos\frac{2\pi}{n})}$ and $\alpha \leq \frac{d_n}{d_n+3}$.

(iii) If $(\boldsymbol{Z}_i, ..., \boldsymbol{Z}_{i-k}) = (\boldsymbol{Z}_j, ..., \boldsymbol{Z}_{j-k})$, then

$$\|\boldsymbol{X}_i - \boldsymbol{X}_j\| \ \leq \ (2\ (d_n + 3)\ \alpha)\ \frac{\beta}{3}\ \alpha^k.$$

(iv) If $(\boldsymbol{Z}_i, ..., \boldsymbol{Z}_{i-(k-1)}) = (\boldsymbol{Z}_j, ..., \boldsymbol{Z}_{j-(k-1)})$,
but $\boldsymbol{Z}_{i-k} \neq \boldsymbol{Z}_{j-k}$, then

$$\|\boldsymbol{X}_i - \boldsymbol{X}_j\| \ \geq \ d_n\ \frac{\beta}{3}\ \alpha^k.$$

Theorem 6.1 (continued):

Let $d_n \ = \ \sqrt{2(1-\cos\frac{2\pi}{n})}$ and $\alpha \leq \frac{d_n}{d_n+3}$.

(v) If even $\alpha < \frac{d_n}{2(d_n+3)}$, then

$$\|\boldsymbol{X}_i - \boldsymbol{X}_j\| \ < \ d_n \, \frac{\beta}{3} \, \alpha^k,$$

if and only if

$$(\boldsymbol{Z}_i, ..., \boldsymbol{Z}_{i-k}) \ = \ (\boldsymbol{Z}_j, ..., \boldsymbol{Z}_{j-k}).$$

Circle Transformations

An Example

Example: Shakespeare's poem ”Venus and Adonis”.

Ignoring capitals, punctuation marks, and blanks, alphabet $\mathcal{A}$ of $n = 26$ letters:

`a, b, c, d, e, f, g, h, i, j, k, l, m, n,`

`o, p, q, r, s, t, u, v, w, x, y, z.`

Text is a series of letters.

Applying *circle transformation* leads to $\boldsymbol{Z}_1, \boldsymbol{Z}_2, \ldots$.

Example: Shakespeare's poem "Venus and Adonis".

(Continued) Given $\boldsymbol{Z}_1, \boldsymbol{Z}_2, \ldots$.

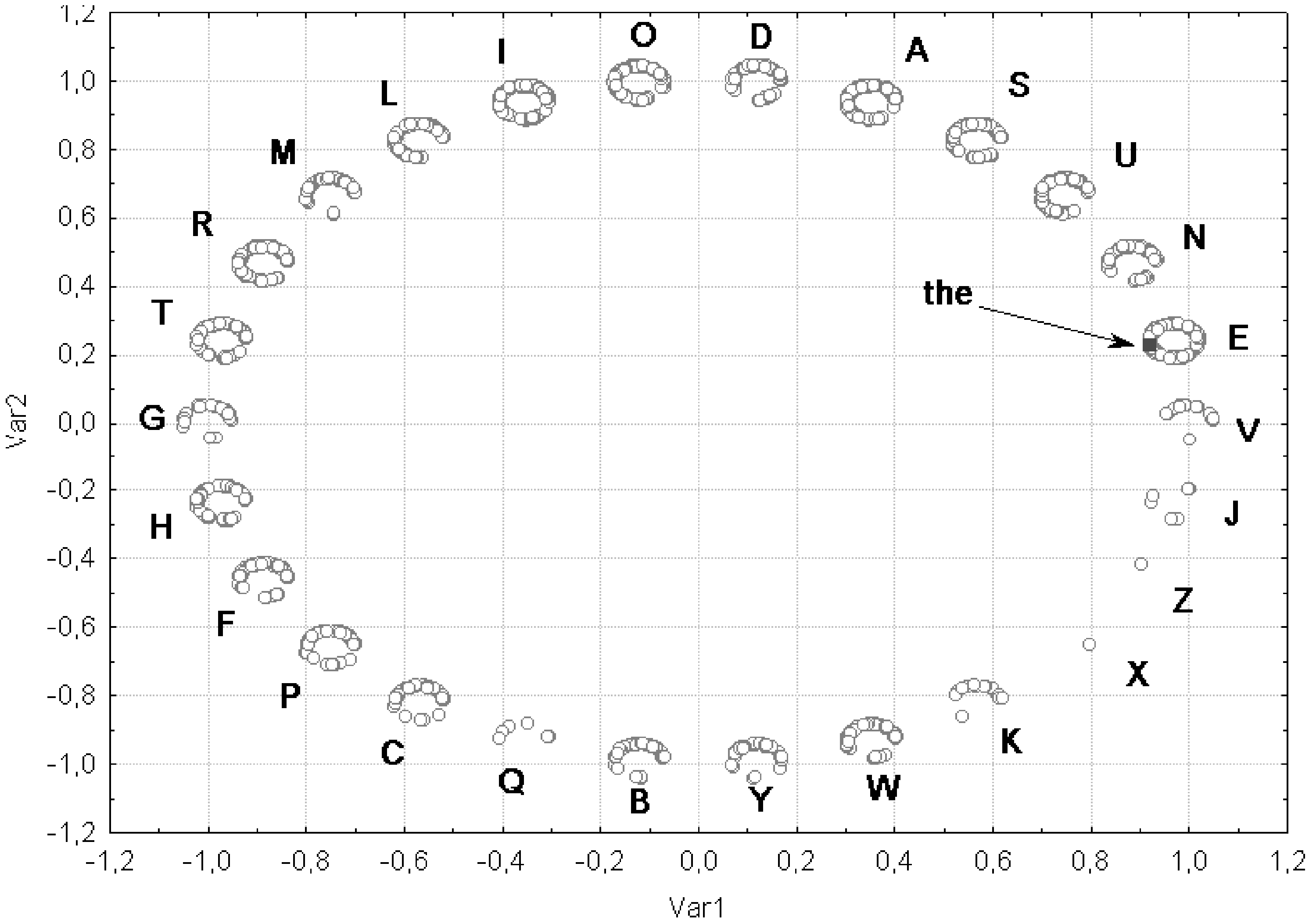
Generation of fractal series $(\boldsymbol{X}_t)_{\mathcal{T}}$ via

$$\boldsymbol{X}_t \;=\; \alpha\boldsymbol{X}_{t-1} \;+\; \beta\boldsymbol{Z}_t, \qquad \text{with } \boldsymbol{X}_0 = \boldsymbol{0},$$

where $\alpha = 0.05$ and $\beta = 1$.

Remark: α even larger than $\frac{d_n}{2(d_n+3)} \approx 0.0372$

$\rightarrow$ pessimistic bound in Theorem 6.1.



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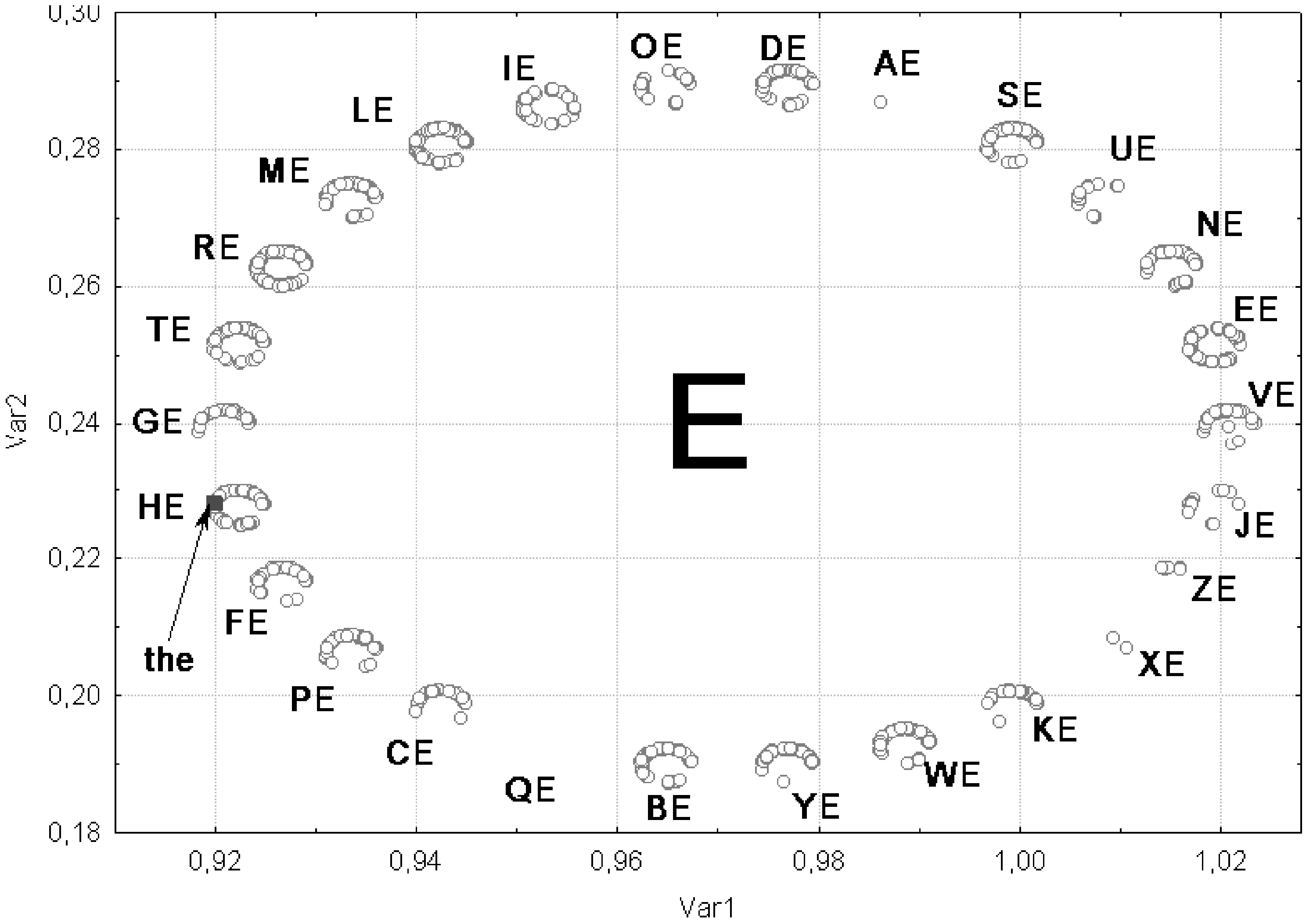
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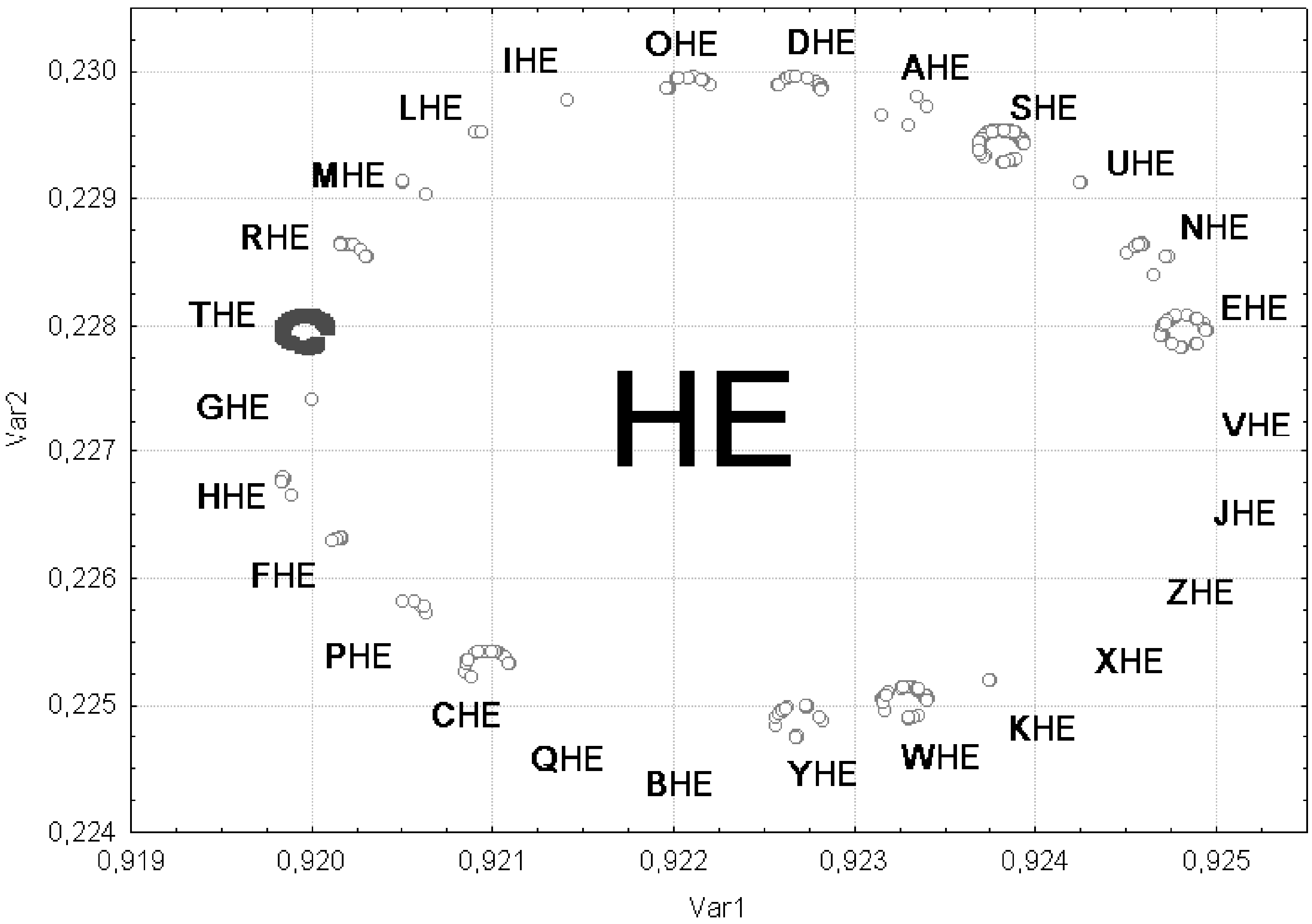

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Thank You
for Your Interest!

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