Bivariate Binomial Autoregressive Models

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This talk is based on the article

Scotto, M.G., Weiß, C.H., Silva, M.E., Pereira, I. (2014).

Bivariate Binomial Autoregressive Models.

Journal of Multivariate Analysis **125**, 233–251.

Further details and references are provided by this article.

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Prerequisites:

Bivariate
Binomial Distributions

Definition & Properties

Bivariate Bernoulli distribution:

(Kocherlakota & Kocherlakota, 1992)

$\boldsymbol{Y} := [Y_1\ Y_2]'$ bivariate Bernoulli random variable,
possible outcomes $(1,1),(1,0),(0,1),(0,0)$
with probabilities $p_{11}, p_{10}, p_{01}, p_{00}$.

p_{ij} determined by parameters
$0 < \alpha_1, \alpha_2 < 1$ and $0 < \alpha < \min\{\alpha_1, \alpha_2\}$:

$$p_{11} = \alpha, \quad p_{11} + p_{10} = \alpha_1, \quad p_{11} + p_{01} = \alpha_2.$$

Marginals Y_i univariately Bernoulli with probability α_i.

Bivariate Bernoulli distribution: (cont.)

(Kocherlakota & Kocherlakota, 1992)

Correlation between Y_1 and Y_2:

$$\rho(Y_1, Y_2) = \frac{\alpha - \alpha_1\alpha_2}{\sqrt{\alpha_1\alpha_2(1-\alpha_1)(1-\alpha_2)}} =: \phi_\alpha.$$

$\Rightarrow$ alternative parametrization with "correlation parameter" ϕ_α.

Range of ϕ_α restricted by (Scotto et al., 2014)

$$\max\left\{-\sqrt{\frac{\alpha_1\alpha_2}{(1-\alpha_1)(1-\alpha_2)}},\ -\sqrt{\frac{(1-\alpha_1)(1-\alpha_2)}{\alpha_1\alpha_2}}\right\} < \phi_\alpha < \min\left\{\sqrt{\frac{\alpha_1(1-\alpha_2)}{(1-\alpha_1)\alpha_2}},\ \sqrt{\frac{(1-\alpha_1)\alpha_2}{\alpha_1(1-\alpha_2)}}\right\}.$$

Bivariate Bernoulli distribution: (cont.)

(Kocherlakota & Kocherlakota, 1992)

Probability generating function (pgf) equals

$$\begin{aligned} G_{\boldsymbol{Y}}(z_1, z_2) \ &:= \ E(z_1^{Y_1} z_2^{Y_2}) \\ &= \ (1-\alpha_1+\alpha_1 z_1)(1-\alpha_2+\alpha_2 z_2) \\ &\quad +\phi_\alpha \sqrt{\alpha_1\alpha_2(1-\alpha_1)(1-\alpha_2)} \cdot (1-z_1)(1-z_2). \end{aligned}$$

So $\phi_\alpha = 0$ implies that Y_1, Y_2 independent Bernoulli r.v.

Construction of

Bivariate binomial distribution of type I:

(Marshall & Olkin, 1985; Kocherlakota & Kocherlakota, 1992)

Let $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_k$ i.i.d. bivariate Bernoulli r.v.

Sample sum $\boldsymbol{W} := \boldsymbol{Y}_1 + \cdots + \boldsymbol{Y}_k$ said to follow

$\text{BVB}_{\text{I}}(k; \alpha_1, \alpha_2, \phi_\alpha)$ or $\text{BVB}_{\text{I}}(k; \alpha_1, \alpha_2, \alpha)$, respectively.

Marginals univariate binomial r.v.: $W_i \sim \text{Bi}(k, \alpha_i)$.

Construction of

Bivariate binomial distribution of type II:

(Marshall & Olkin, 1985; Kocherlakota & Kocherlakota, 1992)

Let $n_1, n_2 \geq 0$ and $0 \leq k \leq \min\{n_1, n_2\}$.

Let $\boldsymbol{W}, U, V$ be independent, where $\boldsymbol{W} \sim \mathrm{BVB}_{\mathrm{I}}(k; \alpha_1, \alpha_2, \phi_\alpha)$,

$U \sim \mathrm{Bi}(n_1 - k, \alpha_1)$ and $V \sim \mathrm{Bi}(n_2 - k, \alpha_2)$.

Then $\boldsymbol{X} := [W_1 + U \;\; W_2 + V]'$ said to follow

$\mathrm{BVB}_{\mathrm{II}}(n_1, n_2, k; \alpha_1, \alpha_2, \alpha)$ or $\mathrm{BVB}_{\mathrm{II}}(n_1, n_2, k; \alpha_1, \alpha_2, \phi_\alpha)$, resp.

Bivariate binomial distribution of type II: (cont.)

(Marshall & Olkin, 1985; Kocherlakota & Kocherlakota, 1992)

Let $\boldsymbol{X} = [X_1 \;\; X_2]' \;\sim\; \text{BVB}_{\text{II}}(n_1, n_2, k; \alpha_1, \alpha_2, \phi_\alpha)$. Then

- $X_i \;\sim\; \text{Bi}(n_i, \alpha_i)$,
- $\rho(X_1, X_2) \;=\; \dfrac{k}{\sqrt{n_1 n_2}} \cdot \dfrac{\alpha - \alpha_1\alpha_2}{\sqrt{\alpha_1\alpha_2(1-\alpha_1)(1-\alpha_2)}} \;=\; \dfrac{k}{\sqrt{n_1 n_2}} \cdot \phi_\alpha$,
- pgf $G_{\boldsymbol{X}}(z_1, z_2)$ equals

$$(1 - \alpha_1 + \alpha_1 z_1)^{n_1 - k}(1 - \alpha_2 + \alpha_2 z_2)^{n_2 - k} \cdot G_{\boldsymbol{Y}}^{k}(z_1, z_2),$$

Bivariate binomial distribution of type II: (cont.)

(Marshall & Olkin, 1985; Kocherlakota & Kocherlakota, 1992)

Let $\boldsymbol{X} = [X_1 \ X_2]' \sim \text{BVB}_{\text{II}}(n_1, n_2, k; \alpha_1, \alpha_2, \phi_\alpha)$. Then

- pmf $p_{(n_1,n_2,k;\alpha_1,\alpha_2,\alpha)}(x_1, x_2) := \text{P}(X_1 = x_1, X_2 = x_2)$

$$= \sum_{j_1=0}^{\min\{x_1,n_1-k\}} \sum_{j_2=0}^{\min\{x_2,n_2-k\}} \binom{n_1-k}{j_1}\binom{n_2-k}{j_2} \alpha_1^{j_1}(1-\alpha_1)^{n_1-k-j_1} \alpha_2^{j_2}(1-\alpha_2)^{n_2-k-j_2}$$

$$\cdot \sum_{i=\max\{0,x_1-j_1+x_2-j_2-k\}}^{\min\{x_1-j_1,x_2-j_2\}} \binom{k}{i,\ x_1-j_1-i,\ x_2-j_2-i,\ k+i+j_1+j_2-x_1-x_2}$$

$$\alpha^i(\alpha_1-\alpha)^{x_1-j_1-i}(\alpha_2-\alpha)^{x_2-j_2-i}(1+\alpha-\alpha_1-\alpha_2)^{k+i+j_1+j_2-x_1-x_2}.$$

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Modelling Bivariate Time Series of Counts:

The BVB_{II}-INARCH(1) Model

Definition & Properties

Motivation: Univariate **binomial INARCH(1) model** by Weiß & Pollett (2014), defined via

$$X_t \,\big|\, X_{t-1} \;\sim\; \text{Bi}\big(n,\ \alpha_0 + \alpha_1\, X_{t-1}/n\big).$$

Properties:

- $\mu \;=\; \dfrac{n\,\alpha_0}{1-\alpha_1}$ and $\sigma^2 \;=\; \mu(1-\frac{\mu}{n})\cdot \underbrace{\dfrac{1}{1-(1-\frac{1}{n})\alpha_1^2}}_{\in\ [1;n)}$,

  i. e., **extra-binomial variation**;

- $E(X_t \mid X_{t-1}) \;=\; \alpha_1\, X_{t-1} + n\,\alpha_0,$

  $V(X_t \mid X_{t-1}) \;=\; n\,\alpha_0\,(1-\alpha_0) - \frac{\alpha_1^2}{n}\, X_{t-1}^2 + (1-2\alpha_0)\,\alpha_1\, X_{t-1};$

- $\rho(k) = \alpha_1^k.$

Definition: Let $\boldsymbol{n} := [n_1 \ n_2]' \in \mathbb{N}^2$ be vector of upper limits for bivariate range.

BVB$_{\text{II}}$-INARCH(1) process $(\boldsymbol{X}_t)$ satisfies

$$\boldsymbol{X}_t \,\Big|\, \boldsymbol{X}_{t-1} \ \sim$$

$$\text{BVB}_{\text{II}}\left(n_1,\, n_2,\, k;\ \alpha_{0,1} + \alpha_{1,1}\,\frac{X_{t-1,1}}{n_1},\ \alpha_{0,2} + \alpha_{1,2}\,\frac{X_{t-1,2}}{n_2},\ \phi\right),$$

where $k := \min\{n_1, n_2\}$, $|\phi| < 1$ and $\alpha_{0,1}, \alpha_{0,2}, \alpha_{0,1} + \alpha_{1,1}, \alpha_{0,2} + \alpha_{1,2} \in (0;1)$.

Marg. $X_{t,i} \quad \sim \quad$ **binom. INARCH(1) model** with $(n_i,\ \alpha_{0,i}, \alpha_{1,i})$.

Parameter ϕ of BVB_{II}-INARCH(1) model has to satisfy

$$\max\left\{-\sqrt{\frac{\alpha_{0,1}\alpha_{0,2}}{(1-\alpha_{0,1})(1-\alpha_{0,2})}},\ -\sqrt{\frac{(1-\alpha_{0,1}-\alpha_{1,1})(1-\alpha_{0,2}-\alpha_{1,2})}{(\alpha_{0,1}+\alpha_{1,1})(\alpha_{0,2}+\alpha_{1,2})}}\right\}$$

$$< \ \phi \ < \ \min\left\{\sqrt{\frac{\alpha_{0,1}(1-\alpha_{0,2}-\alpha_{1,2})}{(1-\alpha_{0,1})(\alpha_{0,2}+\alpha_{1,2})}},\ \sqrt{\frac{(1-\alpha_{0,1}-\alpha_{1,1})\alpha_{0,2}}{(\alpha_{0,1}+\alpha_{1,1})(1-\alpha_{0,2})}}\right\}.$$

Transition probabilities at lag 1,

$$p(\boldsymbol{x}|\boldsymbol{y}) \ = \ p_{(n_1,n_2;\alpha_{0,1}+\alpha_{1,1}\,y_1/n_1,\alpha_{0,2}+\alpha_{1,2}\,y_2/n_2,\phi)}(x_1,x_2),$$

truly positive $\quad\Rightarrow\quad$ primitive and finite-state Markov chain

$\Rightarrow$

irreducible, aperiodic, unique stationary marg. distr.: $\mathbf{Q}\,\boldsymbol{p} = \boldsymbol{p}$.

Theorem:

Marginal moments as for binomial INARCH(1) model,

cross-covariance function takes the form

$$Cov(X_{t,1}, X_{t,2}) \;=\; \frac{k\phi}{1-\alpha_{1,1}\alpha_{1,2}} \cdot h(X_{t,1}, X_{t,2})$$

with $h(X_{t,1}, X_{t,2})$ given by

$$E\left(\sqrt{(\alpha_{0,1}+\alpha_{1,1}\,\frac{X_{t,1}}{n_1})(1-\alpha_{0,1}-\alpha_{1,1}\,\frac{X_{t,1}}{n_1})(\alpha_{0,2}+\alpha_{1,2}\,\frac{X_{t,2}}{n_2})(1-\alpha_{0,2}-\alpha_{1,2}\,\frac{X_{t,2}}{n_2})}\right).$$

Proofs: Scotto et al. (2014).

Numerical illustration: BVB_{II}-INARCH(1) model with $(n_1, n_2, \alpha_{0,1}, \alpha_{1,1}, \alpha_{0,2}, \alpha_{1,2}) = (5, 7, 0.35, 0.3, 0.28, 0.3)$.

Above formula for range of ϕ gives -0.4576 and 0.4576.

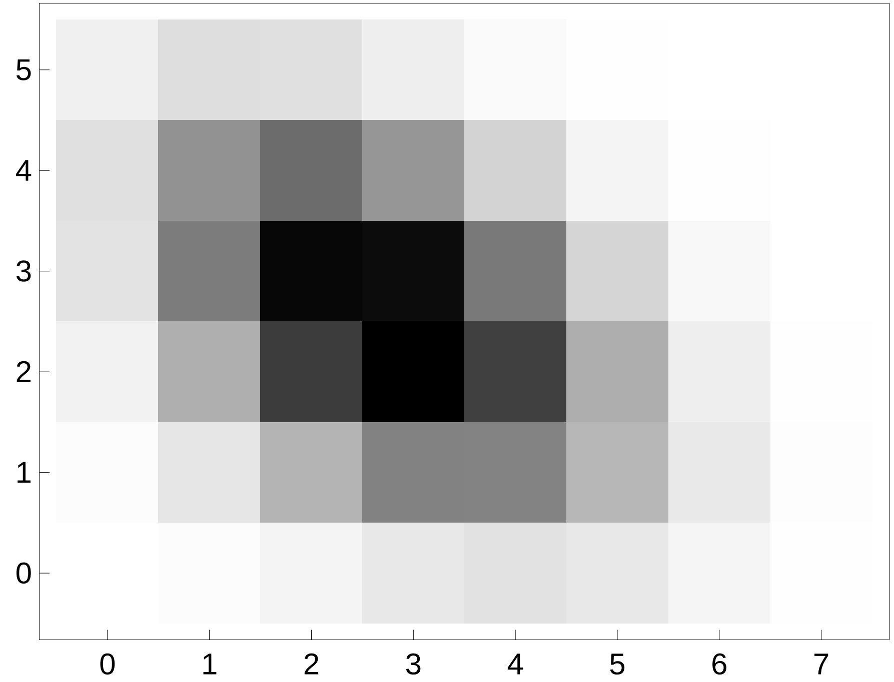
Models from

opposite ends of scale:

$\phi = -0.45$:

cross-covariance -0.595,

cross-correlation -0.380.



Numerical illustration: BVB_{II}-INARCH(1) model with $(n_1, n_2, \alpha_{0,1}, \alpha_{1,1}, \alpha_{0,2}, \alpha_{1,2}) = (5, 7, 0.35, 0.3, 0.28, 0.3)$.

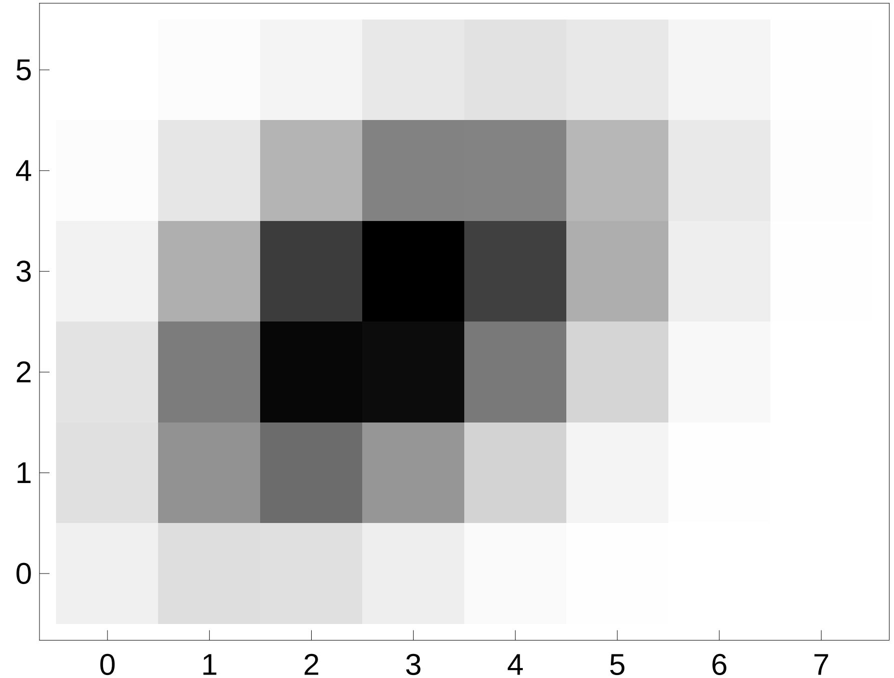
Above formula for range of ϕ gives -0.4576 and 0.4576.

Models from opposite ends of scale:

$\phi = +0.45$:

cross-covariance $+0.595$,

cross-correlation $+0.380$.



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Intermediate Step:

Bivariate Binomial Thinning Operation

Definition & Properties

Motivation: Univariate **binomial thinning operator**:

(Steutel & van Harn, 1979)

Let X be count data r.v., then

$$\alpha \circ X := \sum_{i=1}^{X} Y_i, \qquad \text{where } Y_i \text{ are i.i.d. Bi}(1, \alpha),$$

i. e., $\alpha \circ X \ \sim\ \text{Bi}(X, \alpha)$ and has range $\{0, \ldots, X\}$.

($\approx$ number of "survivors" from population of size X)

Used in count data time series modelling,

replaces multiplications in ARMA recursion.

Definition: Let $\boldsymbol{X} := [X_1\ \ X_2]'$ be bivariate count data r.v., abbreviate $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, \phi_\alpha)$ with $0 < \alpha_1, \alpha_2 < 1$ and ϕ satisfying restriction for bivariate Bernoulli distribution.

Bivariate binomial thinning operation:

$$\boldsymbol{\alpha} \otimes \boldsymbol{X} \,\Big|\, \boldsymbol{X} \sim \text{BVB}_{\text{II}}\left(X_1, X_2, \min\{X_1, X_2\};\ \alpha_1, \alpha_2, \phi_\alpha\right).$$

Behaves marginally as usual binomial thinning:

$$(\boldsymbol{\alpha} \otimes \boldsymbol{X})_i \,\Big|\, \boldsymbol{X} \sim \text{Bi}(X_i, \alpha_i) \qquad \text{for } i = 1, 2.$$

Lemma: Properties of bivariate binomial thinning.

Cross-covariance:

$$Cov\Big((\boldsymbol{\alpha} \otimes \boldsymbol{X})_1,\ (\boldsymbol{\alpha} \otimes \boldsymbol{X})_2\Big) \ =$$

$$\alpha_1\alpha_2 \cdot Cov(X_1, X_2) + \phi_\alpha\sqrt{\alpha_1\alpha_2(1-\alpha_1)(1-\alpha_2)} \cdot E\Big(\min(X_1, X_2)\Big),$$

i. e., causes additional cross-correlation as long as $\phi_\alpha \neq 0$, might be both positive or negative.

Proofs: Scotto et al. (2014).

Possible alternative:

Bivariate matrix thinning: (Franke & Rao, 1993)

$$\mathbf{A} \circ \boldsymbol{X} = \begin{bmatrix} a_{11} \circ X_1 + a_{12} \circ X_2 \\ a_{21} \circ X_1 + a_{22} \circ X_2 \end{bmatrix} \qquad \text{with } \mathbf{A} \in [0;1]^{2\times 2}.$$

But behaves marginally as usual binomial thinning only if $a_{12} = a_{21} = 0$. (Pedeli & Karlis, 2011)

But for diagonal matrix thinning no cross-correlation.

Bivariate binomial thinning combines cross-correlation *plus* being marginally a usual binomial thinning.

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Modelling Bivariate Time Series of Counts:

The $\mathrm{BVB_{II}}$-AR(1) Model

Definition & Properties

Motivation: Univariate **binomial AR(1) model** by McKenzie (1985), defined via

$$X_t = \alpha \circ X_{t-1} + \beta \circ (n - X_{t-1}) \qquad \text{with } \alpha = \beta + \rho,\ \beta = \pi(1-\rho),$$

for $\pi \in (0;1)$ and $\rho \in \big(\max\{-\pi/(1-\pi), -(1-\pi)/\pi\};\ 1\big)$.

Properties:

- Marginals $X_t \sim \text{Bi}(n, \pi)$;
- $E(X_t \mid X_{t-1}) = \rho \cdot X_{t-1} + n\beta$,
  $V(X_t \mid X_{t-1}) = \rho(1-\rho)(1-2p) \cdot X_{t-1} + n\beta(1-\beta)$;
- $\rho(k) = \rho^k$; and many more.

(Weiß & Pollett, 2012; Weiß & Kim, 2013).

Definition: Let $\boldsymbol{n} := [n_1\ n_2]' \in \mathbb{N}^2$ be vector of upper limits for bivariate range.

BVB$_{\mathrm{II}}$-AR(1) process $(\boldsymbol{X}_t)$ satisfies

$$\boldsymbol{X}_t \;=\; \boldsymbol{\alpha} \otimes \boldsymbol{X}_{t-1} \;+\; \boldsymbol{\beta} \otimes (\boldsymbol{n} - \boldsymbol{X}_{t-1}),$$

where $\beta_i := \pi_i \cdot (1-\rho_i)$ and $\alpha_i := \beta_i + \rho_i$,
$\pi_i \in (0;1)$ and $\rho_i \in \big(\max\{-\pi_i/(1-\pi_i), -(1-\pi_i)/\pi_i\};\, 1\big)$.

Marg. $X_{t,i} \;\sim\;$ **binom. AR(1) model** with $(n_i,\ \pi_i, \rho_i)$.

Cross-correlation via ϕ_α (extinction) and ϕ_β (colonization), reflects mutual competition and exchange.

Theorem:

Marginal moments as for binomial AR(1) model,

cross-covariance function takes the form

$$Cov(X_{t,1}, X_{t,2}) \;=$$

$$\tfrac{1}{1-\rho_1\rho_2}\Big(\phi_\alpha\sqrt{\alpha_1\alpha_2(1-\alpha_1)(1-\alpha_2)}\cdot E\big(\min(X_{t,1}, X_{t,2})\big)$$

$$+\ \phi_\beta\sqrt{\beta_1\beta_2(1-\beta_1)(1-\beta_2)}\cdot E\big(\min(n_1-X_{t,1}, n_2-X_{t,2})\big)\Big).$$

Proofs: Scotto et al. (2014).

Transition probabilities at lag 1,

$$p(\boldsymbol{x}|\boldsymbol{y}) \;=\; \sum_{a_1=0}^{\min(x_1,y_1)} \sum_{a_2=0}^{\min(x_2,y_2)}$$

$$p_{(y_1,y_2;\alpha_1,\alpha_2,\phi_\alpha)}(a_1,a_2) \cdot p_{(n_1-y_1,n_2-y_2;\beta_1,\beta_2,\phi_\beta)}(x_1-a_1,x_2-a_2),$$

truly positive $\Rightarrow$ primitive and finite-state Markov chain

$\Rightarrow$

irreducible, aperiodic, unique stationary marg. distr.: $\mathbf{Q}\,\boldsymbol{p} = \boldsymbol{p}$.

Numerical illustration: BVB_{II}-AR(1) model
with $(n_1, n_2, \pi_1, \pi_2, \rho_1, \rho_2) = (5, 7, 0.5, 0.4, 0.3, 0.3)$,
same mean and autocorrelation as above BVB_{II}-INARCH(1).

We have $(\alpha_1, \beta_1) = (0.65, 0.35)$ and $(\alpha_2, \beta_2) = (0.58, 0.28)$,
hence $-0.6244 \leq \phi_\alpha \leq 0.8623$ and $-0.4576 \leq \phi_\beta \leq 0.8498$.

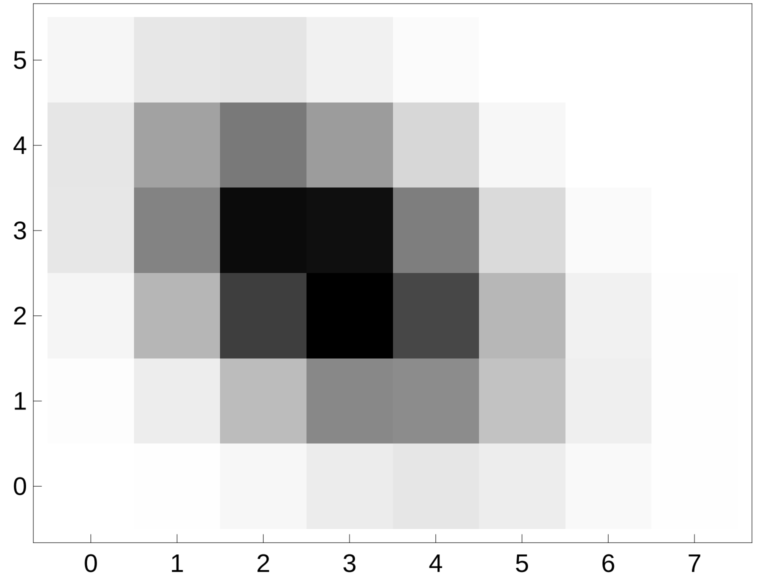
Models from

opposite ends of scale:

$(\phi_\alpha, \phi_\beta) = (-0.62, -0.45)$:

cross-covariance -0.539,

cross-correlation -0.372.



Numerical illustration: BVB_{II}-AR(1) model with $(n_1, n_2, \pi_1, \pi_2, \rho_1, \rho_2) = (5, 7, 0.5, 0.4, 0.3, 0.3)$, same mean and autocorrelation as above BVB_{II}-INARCH(1).

We have $(\alpha_1, \beta_1) = (0.65, 0.35)$ and $(\alpha_2, \beta_2) = (0.58, 0.28)$, hence $-0.6244 \leq \phi_\alpha \leq 0.8623$ and $-0.4576 \leq \phi_\beta \leq 0.8498$.

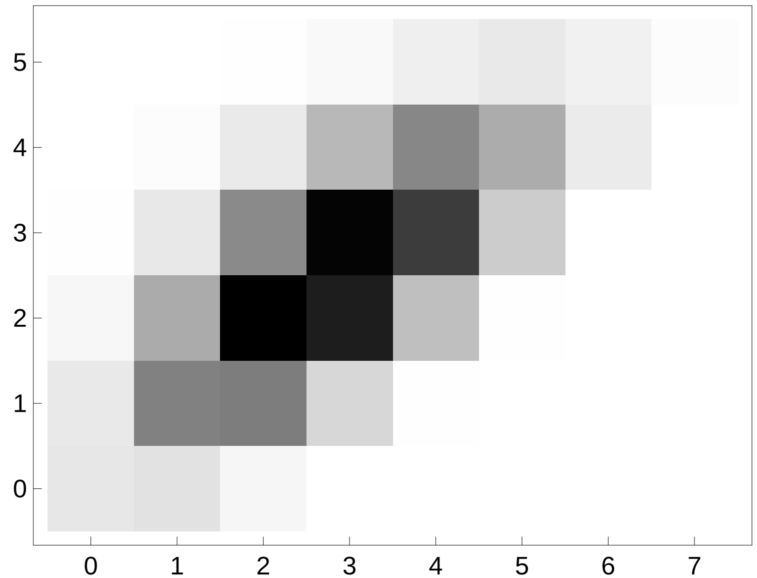
Models from

opposite ends of scale:

$(\phi_\alpha, \phi_\beta) = (0.86, 0.84)$:

cross-covariance $+1.001$,

cross-correlation $+0.691$.



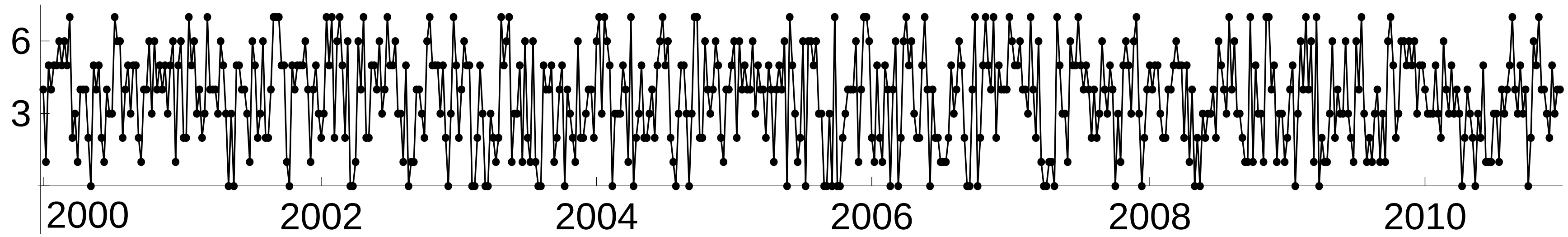
Further Results

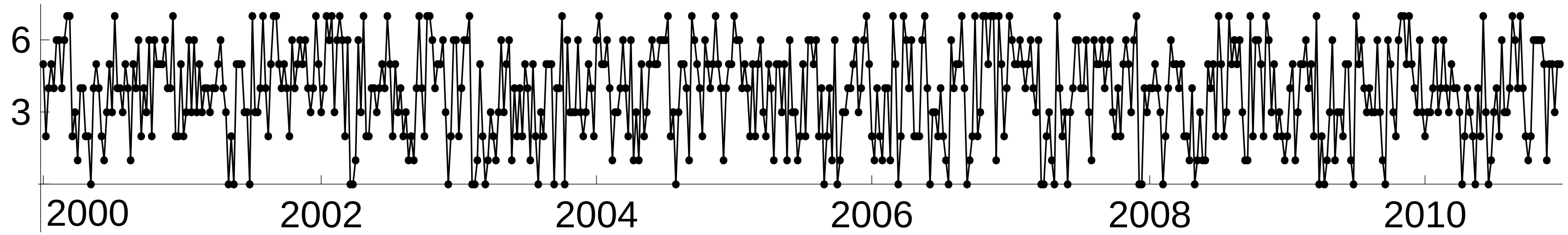
... and Conclusions

In Scotto et al. (2014), we provide detailed treatment of

- **parameter estimation** (conditional maximum likelihood) for both $\mathrm{BVB_{II}}$-INARCH(1) and $\mathrm{BVB_{II}}$-AR(1) model (asymptotic behaviour, finite-sample properties);
- h-step-ahead **forecasting** for $\mathrm{BVB_{II}}$-INARCH(1) and $\mathrm{BVB_{II}}$-AR(1) processes;
- **real-data application:** number of rainy days per week in Bremen and Cuxhaven (DWD, “Deutscher WetterDienst”).

Rainy days in Bremen and Cuxhaven:





$\bar{x}_1 \approx 3.65$ and $\bar{x}_2 \approx 3.84$, $s_1^2 \approx 3.99$ and $s_2^2 \approx 3.88$,
first-order autocorrelation around 0.15–0.20,
cross-correlation around 0.83. $\rightarrow$ BVB_{II}-INARCH(1)

Conclusions

- New bivariate counts time series models with finite range, based on bivariate binomial distribution of type II.

- New bivariate models behave marginally
  like their univariate counterparts,
  allow for both positive and negative cross-correlation.

- Approaches successful for real applications.

- Future research: use bivariate thinning operation for bivariate extension of INAR(1) model,

$$\boldsymbol{X}_t \;=\; \boldsymbol{\alpha} \otimes \boldsymbol{X}_{t-1} \;+\; \boldsymbol{\epsilon}_t.$$

Thank You

for Your Interest!

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