Detecting Mean Increases in Poisson INAR(1) Processes with EWMA Control Charts

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Literature

For references in this talk, see

Weiß(2009): EWMA monitoring of correlated processes of Poisson counts. *QTQM* 6(2), pp. 137-153.

Weiß(2010): Detecting mean increases in Poisson INAR(1) processes with EWMA control charts. Appears in *Journal of Applied Statistics*.

Weiß & Testik(2009): CUSUM monitoring of first-order integer-valued autoregressive processes of Poisson counts. *Journal of Quality Technology* 41(4), pp. 389-400.

Poisson INAR(1) Processes

Definition & Properties

Definition of Poisson INAR(1) process:

Let $(\epsilon_t)_{\mathbb{N}}$ be i.i.d. process with marginal distribution $Po(\mu(1-\alpha))$, where $\mu > 0$ and $\alpha \in (0;1)$. Let $N_0 \sim Po(\mu)$.
If the process $(N_t)_{\mathbb{N}_0}$ satisfies

$$N_t \;=\; \alpha \circ N_{t-1} \;+\; \epsilon_t, \qquad t \geq 1,$$

plus sufficient independence conditions, then it follows a stationary *Poisson INAR(1) model* with marginal distribution $Po(\mu)$.

McKenzie (1985), Al-Osh & Alzaid (1987, 1988)

Binomial thinning, due to Steutel & van Harn (1979):

N discrete random variable with range $\{0, \ldots, n\}$ or $\mathbb{N}_0$.

Binomial thinning

$$\alpha \circ N \;:=\; \sum_{i=1}^{N} X_i,$$

where X_i are independent Bernoulli trials $\sim B(1, \alpha)$.

Guarantees that right-hand side always integer-valued:

$$N_t \;=\; \alpha \circ N_{t-1} \;+\; \epsilon_t.$$

Interpretation: $\alpha \circ N$ is number of survivors.

Basic properties of Poisson INAR(1) processes:

- Stationary Markov chain with $Po(\mu)$-marginals and $p_{k|l} := P(N_t = k \mid N_{t-1} = l) = \sum_{j=0}^{\min(k,l)} \binom{l}{j}\alpha^j(1-\alpha)^{l-j} \cdot e^{-\mu(1-\alpha)}\frac{(\mu(1-\alpha))^{k-j}}{(k-j)!}$,
- autocorrelation $\rho(k) := Corr[N_t, N_{t-k}] = \alpha^k$.

Estimation from time series $N_1, \ldots, N_T$:

$$\hat{\mu} := \frac{1}{T} \cdot \sum_{t=1}^{T} N_t, \qquad \hat{\alpha} = \frac{\sum_{t=2}^{T}(N_t - \bar{N}_T)(N_{t-1} - \bar{N}_T)}{\sum_{t=1}^{T}(N_t - \bar{N}_T)^2}.$$

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Interpretation of INAR(1) process:

$$\underbrace{N_t}_{\text{Population at time } t} \quad = \quad \underbrace{\alpha \circ N_{t-1}}_{\text{Survivors of time } t-1} \quad + \quad \underbrace{\epsilon_t}_{\text{Immigration}}.$$

Interpretation applies well to many real-world problems, e. g.:

- N_t: number of users accessing web server, ϵ_t: number of new users, $\alpha \circ N_{t-1}$: number of previous users still active.
- N_t: number of faults, ϵ_t: number of new faults, $\alpha \circ N_{t-1}$: number of previous faults not rectified yet.

The Poisson INAR(1) model . . .

- is of simple structure,
- essential properties known explicitly,
- is easy to fit to data,
- is easy to interpret,
- applies well to real-world problems, . . .

In a nutshell: A simple model for autocorrelated counts, which is well-suited for SPC!

Controlling Poisson INAR(1) Processes

Control Concepts

Poisson INAR(1) model:

$(N_t)_{\mathbb{N}_0}$ is stationary Poisson INAR(1) process with innovations $(\epsilon_t)_{\mathbb{N}} \sim Po(\mu(1-\alpha))$. So $N_t \sim Po(\mu)$.

State of statistical control: $\mu = \mu_0$ and $\alpha = \alpha_0$.

Weiß (2007) analyzed c-**Chart** for Poisson INAR(1):

+ exact ARLs via Markov chain approach,

+ easily designed and interpreted,

+ effective for very large shifts,

− but very insensitive otherwise!

Weiß (2009) analyzed **Combined EWMA Chart**:

+ exact ARLs via Markov chain approach,

+ applicable for very different types of out-of-control situation,

– difficult to design, six design parameters!

In practice:

Often only interested in detecting **increases in process mean** compared to in-control mean, e. g.,

- counts of defects in manufacturing industry,
- counts of complaints in service industry,
- number of certain infections in epidemiology.

In such situations, combined EWMA chart appears to be overparametrized.

$\rightarrow$ **One-sided CUSUM chart** of Weiß & Testik (2009) attractive alternative:

\+ exact ARLs via Markov chain approach,

\+ easily designed (three design parameters),

\+ very sensitive already to small mean shifts, etc.

$\Rightarrow$ benchmark chart in the following!

Simplified EWMA possible with similar performance?

One-Sided Poisson INAR(1) EWMA Chart

Definition & Properties

One-sided **EWMA chart** for detecting positive shifts in μ:

$$Q_0 = q_0,$$
$$Q_t \;=\; \text{round}\big(\lambda \cdot N_t \;+\; (1-\lambda)\cdot Q_{t-1}\big), \quad t = 1, 2, \ldots$$

$q_0 \geq 0$: starting value, typically $q_0 = 0$.

Fast Initial Response (FIR) feature if $q_0 > 0$.

$\lambda \in (0;1]$: smoothing parameter.

$u > 0$: upper control limit.

$(N_t)_{\mathbb{N}}$ considered in control unless alarm $Q_t \geq u$ triggered.

Properties: One-sided EWMA chart

coincides with combined EWMA chart of Weiß (2009)

if design parameters $l_c = l_e = 0$ and $u_c = \infty$.

$\Rightarrow$ All results of Weiß (2009) directly apply to one-sided EWMA chart.

In particular, ARLs can be computed with MC approach outlined there.

Real-data example of Weiß & Testik (2009):
Time series of counts of IP addresses,
conjectured to stem from following in-control model:
Poisson INAR(1) with $\mu_0 = 1.28$ and $\alpha_0 = 0.29$.

Based on this model, Weiß & Testik (2009) designed
a c chart with UCL $u = 6$ and $ARL_0 \approx 504.949$, and
four CUSUM charts with ARL_0s between 502.586 and 507.447.

Aim: Find EWMA chart design with ARL_0 around 500.

Poisson INAR(1) EWMA – Example

Considered control charts:

- $(u, \lambda, q_0) = (2, 0.11, 1)$ with $ARL_0 \approx 504.949$;
- $(u, \lambda, q_0) = (3, 0.16, 2)$ with $ARL_0 \approx 504.949$;
- $(u, \lambda, q_0) = (4, 0.37, 3)$ with $ARL_0 \approx 592.584$;
- $(u, \lambda, q_0) = (5, 0.63, 1)$ with $ARL_0 \approx 464.239$.

We expect design 4 to show too much false alarms, while design 3 too robust.

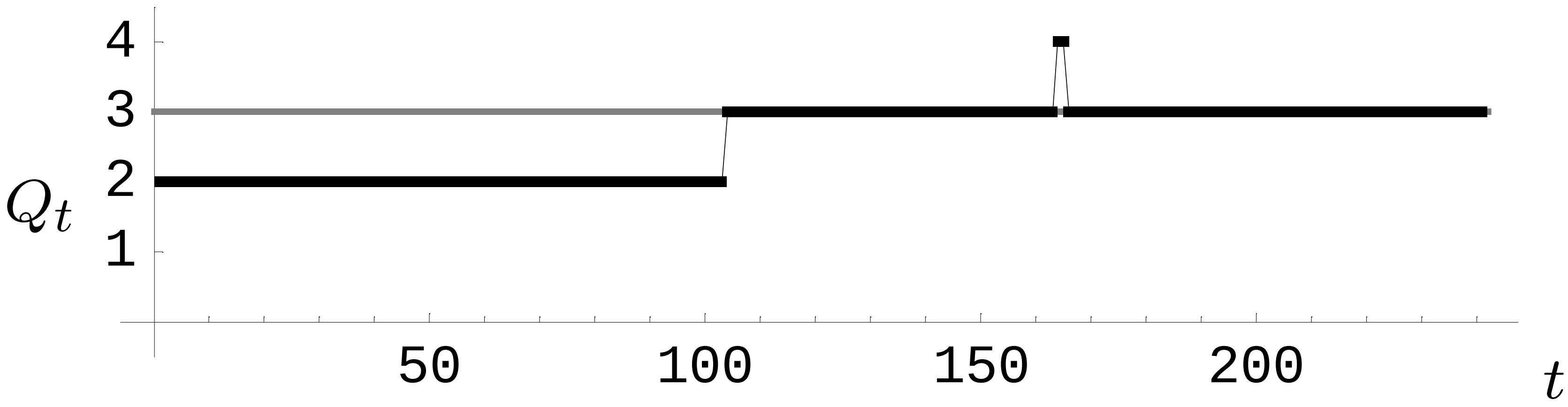
Designs 1 and 2 exactly same ARL_0 as c chart with UCL 6!

Poisson INAR(1) EWMA – Example

In fact, it follows from definition of one-sided EWMA for designs 1 and 2 that statistic Q_t reaches its respective limit u for first time **iff** $N_t = 6$ for first time.

$\Rightarrow$ Both charts lead to equivalent decision rule as c chart.

$\Rightarrow$ Both show same (bad) performance as c chart.



$\Rightarrow$ Due to small value of λ and rounding operation of Q_t, data clearly oversmoothed.

Deficiencies of one-sided EWMA chart:
On one hand, sensitivity becomes better if λ decreases.
On other hand, effect of smoothing increases for decreasing λ.

$\Rightarrow$ Not possible to choose λ as small as perhaps required to reach certain sensitivity.

The s-EWMA Control Chart

Definition & Properties

Basic idea: New rounding operation!

Let $s \in \mathbb{N}$, define $\mathbb{Q}_s := \{\frac{r}{s} \mid r \in \mathbb{Z}\}$ as set of all rationals with denominator s. (Note: $\mathbb{Q}_1 = \mathbb{Z}$)

Define function s-round: $\mathbb{R} \to \mathbb{Q}_s$ by

$$s\text{-round}(x) = z \text{ iff } x \in [z - \frac{1}{2s}; z + \frac{1}{2s}).$$

So s-round maps x onto nearest fraction with denominator s.

Note that 1-round coincides with round.

One-sided s-**EWMA chart** for detecting positive shifts in μ:

$$Q_0 = q_0,$$
$$Q_t \; = \; s\text{-round}\Big(\lambda \cdot N_t \; + \; (1-\lambda) \cdot Q_{t-1}\Big), \quad t = 1, 2, \ldots$$

$(N_t)_{\mathbb{N}}$ considered in control unless alarm $Q_t \geq u$ triggered.

$u \in \mathbb{Q}_s^+$: upper control limit.

$q_0 \in \{0, \ldots, u - \frac{1}{s}\}$: starting value, typically $q_0 = 0$.
Fast Initial Response (FIR) feature if $q_0 > 0$.

$\lambda \in (0; 1]$: smoothing parameter.

s-EWMA Control Chart

Weiß (2010) shows that still $(N_t, Q_t)_{\mathbb{N}}$ is homogeneous bivariate Markov chain.

$\Rightarrow$ Again MC approach for exact ARL computation.

However: Dimension of involved matrices increases with s.

$\Rightarrow$ Values like $s \in \{1, 2, 4\}$ reasonable for practice.

Design of One-Sided s-EWMA Chart:

1. Find best possible 1-EWMA design (u_1, λ_1) according to Weiß (2009),
2. increase s and choose $u \leq u_1$ such that in-control ARL below desired ARL_0,
3. decrease $\lambda \in (0; \lambda_1]$ to adjust in-control ARL close to desired ARL_0,
4. use $q_0 \in \{0, \ldots, u - \frac{1}{s}\}$ for fine-tuning of in-control ARL.

s-EWMA Control Chart – Example

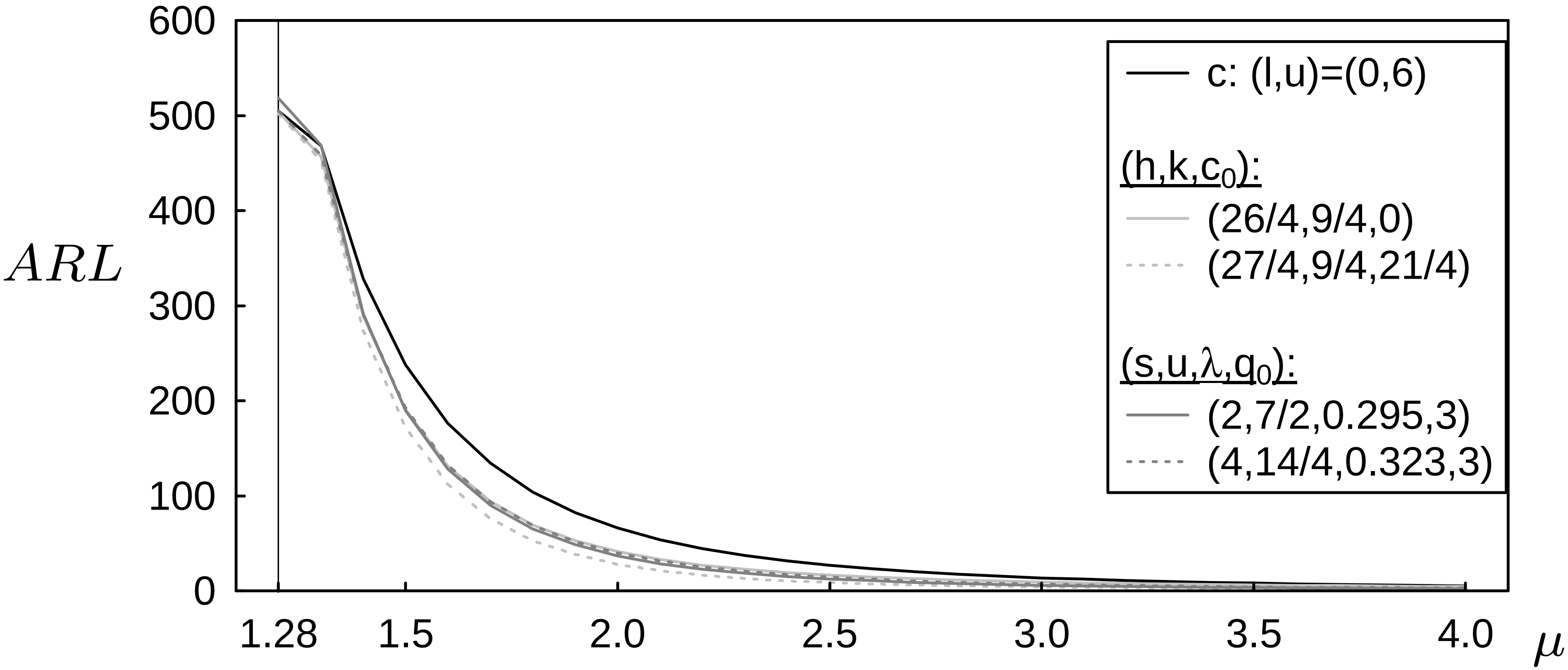
Above real-data example: We find

- 2-EWMA chart with $(u, \lambda, q_0) = (\frac{7}{2}, 0.295, 3)$ and $ARL_0 \approx 518.459$,
- 4-EWMA chart with $(u, \lambda, q_0) = (\frac{14}{4}, 0.323, 3)$ and $ARL_0 \approx 505.301$.

Both charts have similar in-control performance like considered c chart and CUSUM charts.

s-EWMA Control Chart – Example

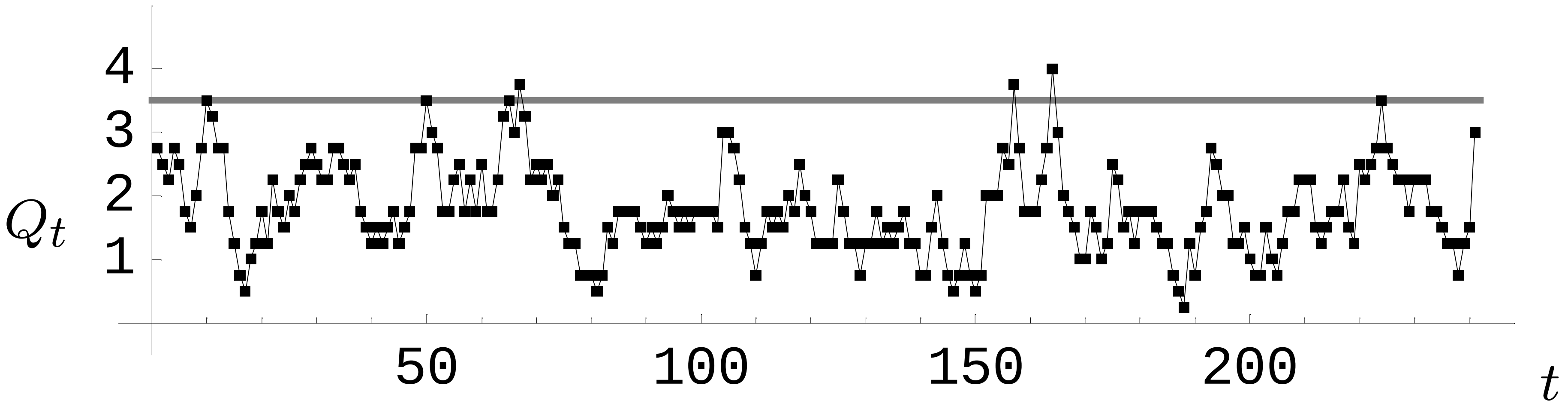
Out-of-control performance of two s-EWMA charts c chart and two most sensitive CUSUM charts



s-EWMA Control Chart – Example

Both EWMA charts noticeably better than c chart, but outperformed by CUSUM chart with FIR.

4-EWMA chart applied to IP data detects the out-of-control situation already at time $t = 10$, like FIR-CUSUM:



s-EWMA Control Chart – Performance

Extensive **performance study** in Weiß (2010):

- s-EWMA chart clearly outperforms c chart,
- best sensitivity for small λ although very large shifts often better detected for large λ,
- choice of $q_0 > \mu_0$ (FIR) slightly improves sensitivity concerning very large shifts ($\geq 100\,\%$),
- but CUSUM shows better performance for small shifts ($5\,\%$ to $20\,\%$); performance of s-EWMA relative to CUSUM better for increasing α_0.

In Weiß (2010), also **robustness against model misspecification** analyzed:

s-EWMA chart based on assumption of INAR(1) process with **Poisson** marginals.
Moderate departures may not be detected.

Most common misspecification: **overdispersion**.
Most popular approaches for modeling overdispersion: negative binomial and generalized Poisson distribution.

Both are possible marginal distributions of INAR(1) model.

s-EWMA Control Chart – Performance

In Weiß (2010), generalized Poisson considered.

s-EWMA chart affected by overdispersion:
in-control performance influenced most severely,
ooc performance for large shifts in μ nearly constant.

Robustness becomes better if λ decreased,
while additional FIR feature does not affect robustness.

For very small λ ($\lambda \approx 0.1$), ARL performance reasonably close to expected one even if up to 20 % overdispersion.

Future Research

Work in progress together with Murat Testik:
Robustness of one-sided CUSUM.

It seems that

- one-sided CUSUM quite sensitive to overdispersion, better robustness for large h, small k, without FIR;
- but a new **Winsorized CUSUM**
  - allows approximate ARL comp. with MC approach,
  - has nearly unaffected ooc performance (Wins. CUSUM **with FIR** even improved ooc perf.),
  - shows a clearly improved robustness.

Thank You

for Your Interest!

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