CUSUM Monitoring of INAR(1) Processes of Poisson Counts

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Some introductory words . . .

For references in this talk, see

CUSUM Monitoring of First-Order Integer-Valued Autoregressive Processes of Poisson Counts.
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Controlling correlated processes of Poisson counts.
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Poisson INAR(1) Processes

Definition & Properties
Definition of Poisson INAR(1) process:

Let \((\epsilon_t)_\mathbb{N}\) be i.i.d. process with marginal distribution \(Po(\lambda(1-\alpha))\), where \(\lambda > 0\) and \(\alpha \in (0; 1)\). Let \(N_0 \sim Po(\lambda)\). If the process \((N_t)_{N_0}\) satisfies

\[
N_t = \alpha \circ N_{t-1} + \epsilon_t, \quad t \geq 1,
\]

plus sufficient independence conditions, then it follows a stationary Poisson INAR(1) model with marginal distribution \(Po(\lambda)\).

Binomial thinning, due to Steutel & van Harn (1979):

$N$ discrete random variable with range $\{0, \ldots, n\}$ or $\mathbb{N}_0$.

**Binomial thinning**

$$\alpha \circ N := \sum_{i=1}^{N} X_i,$$

where $X_i$ are independent Bernoulli trials $\sim B(1, \alpha)$.

Guarantees that right-hand side always integer-valued:

$$N_t = \alpha \circ N_{t-1} + \epsilon_t.$$

**Interpretation:** $\alpha \circ N$ is number of survivors.

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Poisson INAR(1): Dependence & Jumps

Basic properties of Poisson INAR(1) processes:

• Stationary Markov chain with $Po(\lambda)$-marginals and

\[ p_{k|l} := P(N_t = k \mid N_{t-1} = l) = \sum_{j=0}^{\min(k,l)} \binom{l}{j} \alpha^j (1 - \alpha)^{l-j} \cdot e^{-\lambda(1-\alpha)} \frac{(\lambda(1-\alpha))^{k-j}}{(k-j)!}, \]

• autocorrelation $\rho(k) := Corr[N_t, N_{t-k}] = \alpha^k$.

Estimation from time series $N_1, \ldots, N_T$:

\[ \hat{\lambda} := \frac{1}{T} \cdot \sum_{t=1}^{T} N_t, \quad \hat{\alpha} = \frac{\sum_{t=2}^{T} (N_t - \bar{N}_T)(N_{t-1} - \bar{N}_T)}{\sum_{t=1}^{T} (N_t - \bar{N}_T)^2}. \]
Poisson INAR(1) Processes

Interpretation of INAR(1) process:

\[ N_t = \alpha \circ N_{t-1} + \epsilon_t \]

Population at time \( t \) = Survivors of time \( t - 1 \) + Immigration

Interpretation applies well to many real-world problems, e. g.:

- \( N_t \): number of users accessing web server, \( \epsilon_t \): number of new users, \( \alpha \circ N_{t-1} \): number of previous users still active.
- \( N_t \): number of faults, \( \epsilon_t \): number of new faults, \( \alpha \circ N_{t-1} \): number of previous faults not rectified yet.

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The Poisson INAR(1) model . . .

- is of simple structure,
- essential properties known explicitly,
- is easy to fit to data,
- is easy to interpret,
- applies well to real-world problems, . . .

In a nutshell: A simple model for autocorrelated counts, which is well-suited for SPC!

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Controlling Poisson INAR(1) Processes

Control Concepts
Poisson INAR(1) model:

$(N_t)_{N_0}$ is stationary Poisson INAR(1) process with innovations $(\epsilon_t)_N \sim Po(\lambda(1 - \alpha))$. So $N_t \sim Po(\lambda)$.

State of statistical control: $\lambda = \lambda_0$ and $\alpha = \alpha_0$. 
Weiß (2007) proposed the following control charts:

- \(c\)-Chart for Poisson INAR(1),
- Residual control chart,
- Conditional control chart,
- Moving average control chart.

Simulation study for \(ARL\) performance.
Disadvantages of the charts proposed by Weiß (2007):

- Exact $ARL$s are extremely difficult to obtain
  $\Rightarrow$ design difficult;

- not very effective in detecting small to moderate shifts in process mean $\lambda$;

- completely insensitive to an increase in autocorrelation $\alpha$ if process mean $\lambda$ does not change.

Therefore, . . .
Poisson INAR(1)
CUSUM Chart

Definition & Properties
One-sided **CUSUM chart** for detecting positive shifts in $\lambda$:

\[
C_0 = c_0, \\
C_t = \max(0; N_t - k + C_{t-1}), \quad t = 1, 2, \ldots
\]

$c_0 \geq 0$: starting value, typically $c_0 = 0$.

Fast Initial Response (FIR) feature if $c_0 > 0$.

$k \geq \lambda_0$: reference value.

$h > 0$: upper control limit.

$(N_t)_{\mathbb{N}}$ considered in control unless alarm $C_t \geq h$ triggered.
(N_t)_N itself Markov chain ⇒ (C_t)_N not Markovian.

But (N_t, C_t)_N Markov chain with transition probabilities

\[ p(a, b|c, d) = P(N_t = a, C_t = b \mid N_{t-1} = c, C_{t-1} = d) \]
\[ = \delta_{b, \max(0; a-k+d)} \cdot p_{a|c}, \]

\[ p_1(a, b|c) = P(N_1 = a, C_1 = b \mid C_0 = c) \]
\[ = \delta_{b, \max(0; a-k+c)} \cdot p_a. \]

⇒ Adapt **Markov chain approach** of Brook & Evans (1972) for ARL computation.

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ARL Computation of One-sided CUSUM Chart:

$\mathcal{I}(h, k)$: Set of reachable in-control values of $(N_t, C_t)$. Let $\mu_{m,a}$ be expected number of in-control signals before first alarm, given that $(N_1, C_1) = (m, a) \in \mathcal{I}(h, k)$. Define

\[
\begin{align*}
\mu &:= (\ldots, \mu_{n,i}, \ldots)^\top, \\
Q^\top &:= (p(n,i \mid m,a))_{(n,i),(m,a) \in \mathcal{I}(h,k)}.
\end{align*}
\]

Dimension of $Q$ and $\mu$ equals $|\mathcal{I}(h,k)|$.

Then $\mu$ solution of linear equation $(I - Q) \cdot \mu = 1$, and

\[
\text{ARL}(c_0) = 1 + \sum_{(m,a) \in \mathcal{I}(h,k)} \mu_{m,a} \cdot p_1(m,a \mid c_0).
\]
Important issue to speed up ARL computations:
Set $\mathcal{I}$ of reachable in-control values of $(N_t, C_t)$. $|\mathcal{I}|$ determines dimension of matrix $Q$ for Markov chain approach.

Consider case $h, k, c_0 \in \mathbb{N}_0$. First idea: $\mathcal{I} = \mathbb{N}_0 \times \{0, \ldots, h-1\}$?

But $C_t \geq h$ iff $N_t - k + C_{t-1} \geq h$.

So $N_t \geq k + h$ always leads to alarm.

Considering further restrictions leads to . . .
\[ I(h, k) := \{(n, i) \mid 0 \leq i \leq h - 1, \max(0; i + k - h + 1) \leq n \leq i + k\}, \]
which is of size
\[ |I(h, k)| = \frac{1}{2}(h - k)(h + k + 1) + hk. \]

Above arguments can also be applied if \( h, k, c_0 \) take values from \( \{\frac{r}{s} \mid r \in \mathbb{N}_0\} \), where common denominator \( s \in \mathbb{N} \) larger than 1.
Poisson INAR(1)
CUSUM Chart

Performance & Design
Implementation of MC approach in Matlab.

Tables for in-control $\text{ARL}_0$ about 500, in-control mean values $\lambda_0 = 2.5, 5, \text{ and } 10$, in-control dependence values $\alpha_0 = 0.25, 0.50, \text{ and } 0.75$, with and without FIR.

Positive shifts in both $\lambda$ and $\alpha$. 

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ARL performance of CUSUM charts for \((\lambda_0, \alpha_0) = (10, 0.5)\) to detect an increase in \(\lambda\):
ARL performance of $c$ chart, moving average charts with window length $w$ (Weiß, 2007) and CUSUM charts with design triples $(h, k, c_0)$ for $(\lambda_0, \alpha_0) = (8, 0.5)$:
ARL performance of CUSUM charts for $\lambda_0, \alpha_0 = (10, 0.5)$ to detect an increase in $\alpha$.

![Graph showing ARL performance of CUSUM charts for different configurations](image)

- (h,k,c₀): (46,11,0)
- (28,12,0)
- (19,13,0)
- (47,11,21)
Summary:
CUSUM very effective for small to moderate shifts in $\lambda$.
Also sensitive to shifts in $\alpha$.

In addition:
Better sensitivity than CUSUM based on residuals.

Design recommendations:
Choose $k$ as $\lceil \lambda_0 + 1 \rceil$.
Additional FIR feature further improves out-of-control performance.
Poisson INAR(1) CUSUM Chart

Real-Data Example
Weiß (2007): counts of accesses to Statistics web server. Each count represents number of different IP addresses (≈ different users) registered within periods of 2-min length.

IP data between 10 a.m. and 6 p.m. on 29.11.2005: Poisson INAR(1) model with $\lambda = 1.28$ and $\alpha = 0.29$.

$\Rightarrow$ Now 241 counts from 6.12.2005, 10 a.m. to 6 p.m.: In-control model with $\lambda_0 = 1.28$ and $\alpha_0 = 0.29$. 

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Considered control charts:

- $c$-chart, LCL 0 and UCL 6, with $ARL_0 = 504.949$,
- $(h, k, c_0) = (4, 3, 0)$ with $ARL_0 = 506.915$,
- $(h, k, c_0) = (\frac{11}{2}, \frac{5}{2}, 0)$ with $ARL_0 = 507.447$,
- $(h, k, c_0) = (\frac{26}{4}, \frac{9}{4}, 0)$ with $ARL_0 = 503.867$, and
- $(h, k, c_0) = (\frac{27}{4}, \frac{9}{4}, \frac{21}{4})$ (FIR feature) with $ARL_0 = 502.586$. 
ARL performance of $c$ and CUSUM charts for $(\lambda_0, \alpha_0) = (1.28, 0.29)$ to detect an increase in $\lambda$: 

![Graph showing ARL performance for different values of $\lambda$. The graph includes multiple lines representing different parameters for CUSUM charts and a single line for the $c$ chart. The parameters for CUSUM charts include $(4,3,0)$, $(11/2,5/2,0)$, $(26/4,9/4,0)$, and $(27/4,9/4,21/4)$. The ARL values are shown against different values of $\lambda$.]

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After removing an outlier, data seems in control. E. g., CUSUM chart with design \((h, k, c_0) = \left(\frac{27}{4}, \frac{9}{4}, \frac{21}{4}\right)\):
Next, we generated i.i.d. Poisson errors with mean 0.72 and added these counts to the corrected IP data

⇒ shifted IP data, increased Poisson mean 2.

We applied above control charts to the shifted IP data: ...
$c$ chart of shifted IP data:

Alarm at time $t = 104$ ($n_{104} = 6$).
CUSUM chart of shifted IP data with \((h, k, c_0) = (4, 3, 0)\):

Alarm at time \(t = 50\) (\(c_{50} = 4\)).

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CUSUM chart of shifted IP data with \((h, k, c_0) = \left(\frac{11}{2}, \frac{5}{2}, 0\right):\)

Alarm at time \(t = 50\) \((c_{50} = \frac{11}{2})\).
CUSUM chart of shifted IP data with \((h, k, c_0) = \left(\frac{26}{4}, \frac{9}{4}, 0\right)\):

Alarm at time \(t = 13\) \((c_{13} = \frac{26}{4})\).
CUSUM chart of shifted IP data with $(h, k, c_0) = \left(\frac{27}{4}, \frac{9}{4}, \frac{21}{4}\right)$:

![CUSUM chart](image-url)

Alarm at time $t = 10$ ($c_{10} = \frac{31}{4}$).
Conclusions

- **INAR(1) model:**
  Simple, easily interpretable model, well-suited for real-world problems from SPC.

- **CUSUM Chart:**
  Exact $ARL$ computation with Markov chain approach, easy to design (only three design parameters, small $k$, further improvement through FIR), very sensitive to small to moderate shifts in $\lambda$, sensitive to shifts in $\alpha$.

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Thank You for Your Interest!

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