A New Class of Autoregressive Models for Time Series of Binomial Counts

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This talk is based on the paper

Weiß, C. H.:
A New Class of Autoregressive Models for Time Series of Binomial Counts.
Preprint 279, Mathematische Institute der Julius-Maximilians-Universität Würzburg, 2007.

All references mentioned in this talk correspond to the references in this article.

Binomial Thinning and the INAR(1) Model

Introduction

INAR(1) model for processes of counts:

Let $(\epsilon_t)_{\mathbb{N}}$ be i.i.d. process with range $\mathbb{N}_0$, let $\alpha \in [0;1]$. An INAR(1) process $(N_t)_{\mathbb{N}_0}$ follows the recursion

$$N_t \;=\; \alpha \circ N_{t-1} \;+\; \epsilon_t, \qquad t \geq 1.$$

McKenzie (1985), Al-Osh & Alzaid (1987, 1988)

Binomial thinning, due to Steutel & van Harn (1979):

N discrete random variable with range $\{0, \ldots, n\}$ or $\mathbb{N}_0$.
Define random variable

$$\alpha \circ N \; := \; \sum_{i=1}^{N} X_i,$$

where X_i are independent Bernoulli trials, $B(1, \alpha)$, also independent of N $\rightarrow$ *counting series*.

We say: $\alpha \circ N$ arises from N by *binomial thinning*
'$\circ$' is called *binomial thinning operator*.

Interpretation of $\alpha \circ N$:

- Population of size N at a certain time t.

- Later at time $t+1$: population shrinked, because some individuals died.

- Assume that individuals die independently of each other with probability $1-\alpha$

$\Rightarrow$ *Number of survivors* is given by $\alpha \circ N$.

The INAR(1) process . . .

- is easy to interpret,
- is well-suited for many popular count distributions: Poisson, negative binomial, generalized Poisson,
- applies well to typical tasks of SQC,
- can be controlled efficiently, . . .

For details, see

Weiß, C.H.: *Controlling correlated processes of Poisson counts*. QREI 23(6), pp. 741–754, 2007.

. . . but by definition

$$N_t \;=\; \alpha \circ N_{t-1} \;+\; \epsilon_t, \qquad t \geq 1.$$

of the INAR(1) process, the INAR(1) model can be applied to processes of counts with the infinite range $\mathbb{N}_0$ only!

The
Binomial AR(1) Model

Definition & Properties

Binomial AR(1) Model

Let $n \in \mathbb{N}$, $\pi \in (0;1)$ and $\rho \in [\max\left(-\frac{\pi}{1-\pi}, -\frac{1-\pi}{\pi}\right); 1]$.

Define $\beta := \pi \cdot (1-\rho)$ and $\alpha := \beta + \rho$.

The process $(X_t)_{\mathbb{N}_0}$ with

$$X_t \;=\; \alpha \circ X_{t-1} \;+\; \beta \circ (n - X_{t-1}), \qquad t \geq 1, \qquad\qquad X_0 \sim B(n, \pi),$$

where all thinnings are performed independently of each other, and the thinnings at time t are independent of $(X_s)_{s<t}$, is called a **binomial AR(1) process**.

McKenzie (1985)

Interpretation of $X_t = \alpha \circ X_{t-1} + \beta \circ (n - X_{t-1})$:

System of n independent units, either in state 1 or state 0.

X_{t-1}: number of units in state 1 at time $t-1$.

$\alpha \circ X_{t-1}$: number of units still in state 1 at time t, with individual transition probability α.

$\beta \circ (n - X_{t-1})$: number of units, which moved from state 0 to state 1 at time t, with individual transition probability β.

Examples: $$X_t = \alpha \circ X_{t-1} + \beta \circ (n - X_{t-1})$$

- Computer pool with n machines, either occupied (state 1) or not (state 0). Here, X_t is number of machines occupied at time t, consisting of machines occupied before, and machines newly occupied.
- Hotel rooms in certain hotel being occupied at day t ...
- Clerks in a counter room serving a customer ...
- Telephones in a call centre being occupied, etc.

Binomial AR(1) Model

Let $(X_t)_{\mathbb{N}_0}$ be binomial AR(1) process.

- $(X_t)_{\mathbb{N}_0}$ is a stationary Markov chain with marginal distribution $B(n,\pi)$

- transition probabilities
  $p_{k|l} := P(X_t = k \mid X_{t-1} = l) \; = \; \sum_{m=\max(0,k+l-n)}^{\min(k,l)}$

$$\binom{l}{m} \binom{n-l}{k-m} \alpha^m (1-\alpha)^{l-m} \, \beta^{k-m} (1-\beta)^{n-l+m-k}.$$

(. . .)

- autocorrelation function

$$\rho(k) := Corr[X_t, X_{t-k}] \;=\; \rho^k, \qquad k \geq 0$$

- conditional moments:

$$E[X_t \mid X_{t-1}] \;=\; \rho \cdot X_{t-1} \;+\; n\beta, \qquad \text{and}$$
$$V[X_t \mid X_{t-1}] \;=\; \rho(1-\rho)(1-2\pi) \cdot X_{t-1} \;+\; n\beta(1-\beta).$$

In a nutshell:

The binomial AR(1) model . . .

- is easy to interpret,
- applies well to typical tasks of SQC,
- essential properties are explicitly known,

but it is only able to model first order dependence, which is too restrictive for practice!

Therefore . . .

The New Family of Binomial AR(p) Models

Definition & Properties

Aim: Extension of the Binomial AR(1) Model to full pth order autoregressive model.

Basic idea:

Adapt multinomial decisions of Lawrance & Lewis (1980):

$\boldsymbol{D} = (D_1, \ldots, D_p) \sim MULT(1; \phi_1, \ldots, \phi_p)$

$X \;:=\; \sum_{k=1}^{p}\, D_k \cdot Z_k \qquad \Rightarrow$

X identical to Z_1 with probability ϕ_1, . . . ,

X identical to Z_p with probability ϕ_p

Definition: As before: $n \in \mathbb{N}$, $\pi \in (0;1)$,
$\rho \in [\max(-\frac{\pi}{1-\pi}, -\frac{1-\pi}{\pi}); 1]$, $\beta := \pi \cdot (1-\rho)$, $\alpha := \beta + \rho$.

$(\boldsymbol{D}_t)_{\mathbb{Z}}$ i.i.d. 'decision' variables:
$\boldsymbol{D}_t = (D_{t,1}, \ldots, D_{t,p}) \sim MULT(1; \phi_1, \ldots, \phi_p)$.

Binomial AR(p) process $(X_t)_{\mathbb{Z}}$ with range $\{0, \ldots, n\}$:

$$X_t \;=\; \sum_{i=1}^{p} \; D_{t,i} \cdot \big(\alpha \circ_t X_{t-i} + \beta \circ_t (n - X_{t-i})\big),$$

plus necessary independence assumptions concerning $\boldsymbol{D}_t$, X_s, $\alpha \circ_{s+j} X_s \; + \; \beta \circ_{s+j} (n - X_s)$ with $s < t$, $j = 1, \ldots, p$.

Equivalently:

$$X_t = \begin{cases} \alpha \circ_t X_{t-1} + \beta \circ_t (n - X_{t-1}) & \text{with prob. } \phi_1, \\ \vdots & \\ \alpha \circ_t X_{t-p} + \beta \circ_t (n - X_{t-p}) & \text{with prob. } \phi_p. \end{cases}$$

Binomial AR(p) Model $\approx$

probabilistic mixture of lagged binomial AR(1) models.

Case of stationarity: Marginal distribution $B(n, \pi)$.

Equivalently:

$$X_t = \begin{cases} \alpha \circ_t X_{t-1} + \beta \circ_t (n - X_{t-1}) & \text{with prob. } \phi_1, \\ \vdots & \\ \alpha \circ_t X_{t-p} + \beta \circ_t (n - X_{t-p}) & \text{with prob. } \phi_p. \end{cases}$$

Binomial AR(p) Model not uniquely determined by above definition: Time index t below thinning '$\circ_t$' indicates that each X_s is involved in thinnings at times $s+1, \ldots, s+p$.

Corresponding joint distribution has to be specified!

General Result on Autocovariance Structure:

Let $(X_t)_{\mathbb{Z}}$ be stationary binomial AR(p) process with marginal distribution $B(n, \pi)$.

Let $\gamma(k) := Cov[X_t, X_{t-k}]$ denote the **autocovariance function**.

Define for $i, k \geq 1$

$$\mu(i,k) \;:=\; E\big[\big(\alpha \circ_t X_{t-i} \;+\; \beta \circ_t (n - X_{t-i})\big) \cdot X_{t-k}\big] \\ \qquad - \rho \cdot E[X_{t-i} \cdot X_{t-k}] \;-\; (1-\rho) \cdot \mu_X^2.$$

...

Autocovariance Structure (continued):

... Then

$$\gamma(k) \;=\; \rho\cdot \textstyle\sum_{i=1}^{p}\ \phi_i\cdot\gamma(|k-i|) \;+\; \textstyle\sum_{i=k+1}^{p}\ \phi_i\cdot\mu(i,k),$$

where $\mu(i,k)=0$ for $i\leq k$, and otherwise

$$\begin{aligned}\mu(i,k) \;=\; \phi_{i-k}\cdot Cov[\alpha\circ_t X_{t-i} \;+\; \beta\circ_t (n-X_{t-i}),\\ \alpha\circ_{t-k} X_{t-i} \;+\; \beta\circ_{t-k} (n-X_{t-i})]\\ -\,\phi_{i-k}\cdot\rho^2\sigma_X^2 \;+\; \rho\cdot\textstyle\sum_{r=k+1}^{i-1}\ \phi_{r-k}\cdot\mu(i,r).\end{aligned}$$

First Special Case: Binomial AR(p) – Identical Thinnings

Definition & Properties

Idea:

Quite intuitive approach is to assume that all thinnings applied to X_t are identical, i. e., each X_t is thinned only once:

$$\alpha \circ_{t+1} X_t \ + \ \beta \circ_{t+1} (n - X_t) \ = \ \ldots$$

$$\alpha \circ_{t+p} X_t \ + \ \beta \circ_{t+p} (n - X_t) \ = \ \alpha \circ X_t \ + \ \beta \circ (n - X_t)$$

Resulting model refered to as **Identical Thinnings** model.

Then

$$Cov[\alpha \circ_t X_{t-i} \;+\; \beta \circ_t (n - X_{t-i}),$$
$$\alpha \circ_{t-k} X_{t-i} \;+\; \beta \circ_{t-k} (n - X_{t-i})] \;=\; \sigma_X^2.$$

Recursion for $\mu(i,k)$, $i > k$, simplifies to

$$\mu(i,k) \;=\; \phi_{i-k} \cdot (1 - \rho^2) \cdot \sigma_X^2 \;+\; \rho \cdot \sum_{r=k+1}^{i-1} \phi_{r-k} \cdot \mu(i,r).$$

Nevertheless, still autocorrelation structure similar to that of an ARMA($p, p-1$) model.

Restricted practical use!

Second Special Case: Binomial AR(p) – Independent Thinnings

Definition & Properties

Idea:

All thinnings are performed independently of each other, i. e., at each time $t+j$, $j=1,\ldots,p$, X_t is newly involved in thinning operations, disregarding the result of previous thinnings.

Consequence: All thinnings $\alpha \circ_{t+j} X_t \; + \; \beta \circ_{t+j} (n-X_t)$ conditionally independent, conditioned on X_t. Resulting model refered to as **Independent Thinnings** model.

Then

$$Cov[\alpha \circ_t X_{t-i} \; + \; \beta \circ_t (n - X_{t-i}), \\ \alpha \circ_{t-k} X_{t-i} \; + \; \beta \circ_{t-k} (n - X_{t-i})] \; = \; \rho^2 \cdot \sigma_X^2.$$

Recursion for $\mu(i,k)$, $i > k$, results in $\mu(i,k) = 0$ for all $i > k$.

Therefore,

$$\rho(k) \; = \; \rho \cdot \sum_{i=1}^{p} \; \phi_i \cdot \rho(|k-i|).$$

Hence, this type of binomial AR(p) model has an **AR(p)-like autocorrelation structure**!

Attractive properties:

Model order p via **partial autocorrelation function**.

Conditional distribution:

$$P(X_t = x \mid X_{t-1} = x_{t-1}, X_{t-2} = x_{t-2}, \ldots)$$

$$= \sum_{i=1}^{p} \phi_i \cdot \sum_{y=0}^{x} \binom{x_{t-i}}{y} \alpha^y (1-\alpha)^{x_{t-i}-y} \cdot \binom{n-x_{t-i}}{x-y} \beta^{x-y} (1-\beta)^{n-x_{t-i}-x+y}.$$

Conditional expectation:

$$E[X_t \mid X_{t-1}, X_{t-2}, \ldots] = \mu_X \cdot (1-\rho) + \rho \cdot \sum_{i=1}^{p} \phi_i \cdot X_{t-i}.$$

Application for model estimation:

- Yule-Walker estimation.
- Conditional least squares estimation.
- Conditioned maximum likelihood estimation.

Modelling Access Counts

An Example

Example: Access Counts

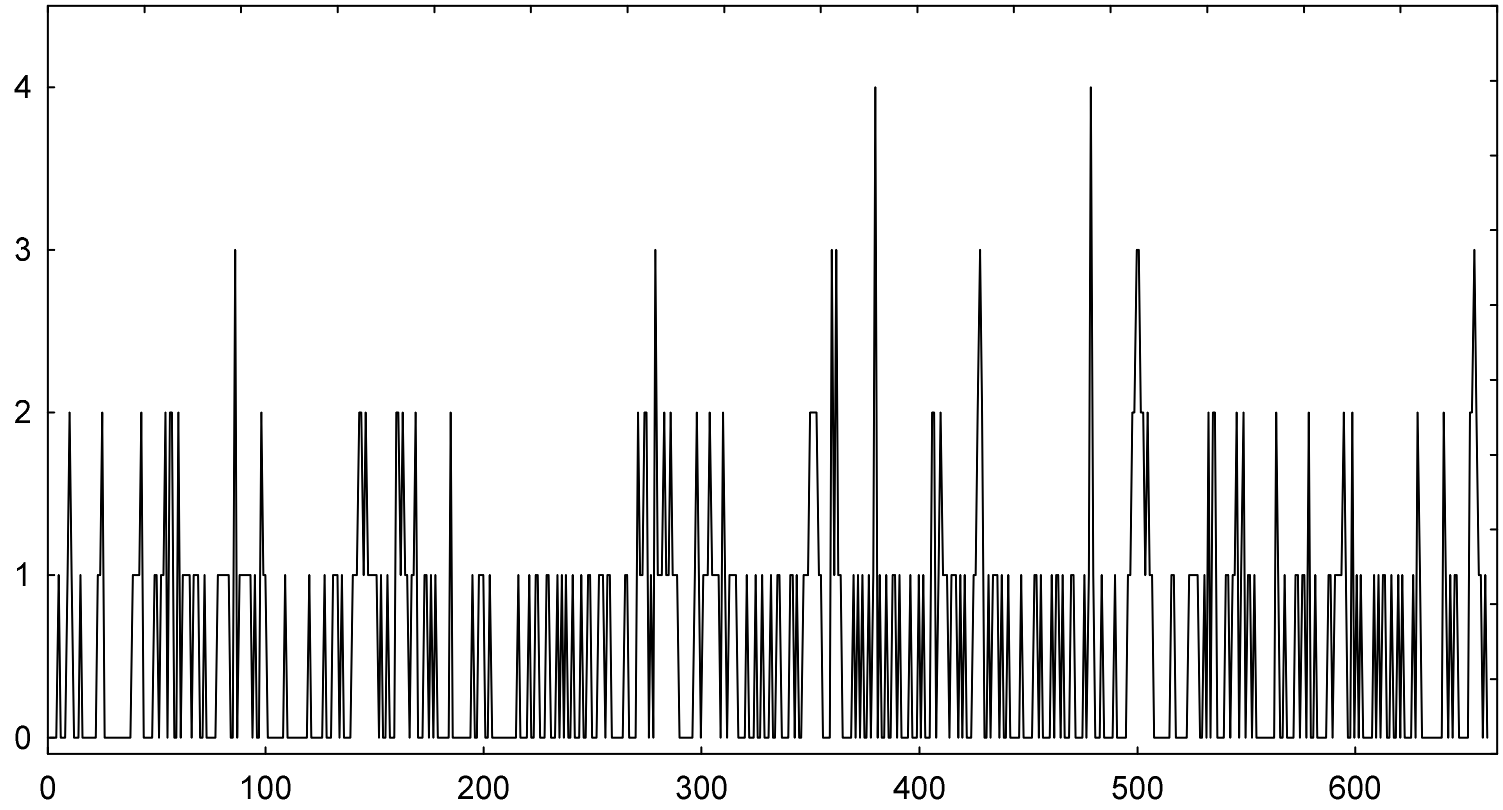
Log data of server of Department of Statistics, University of Würzburg.

From this data, we computed X_t: number of six different staff members, whose home directory was accessed in minute t.

Obviously, X_t has range $\{0, \ldots, 6\}$.

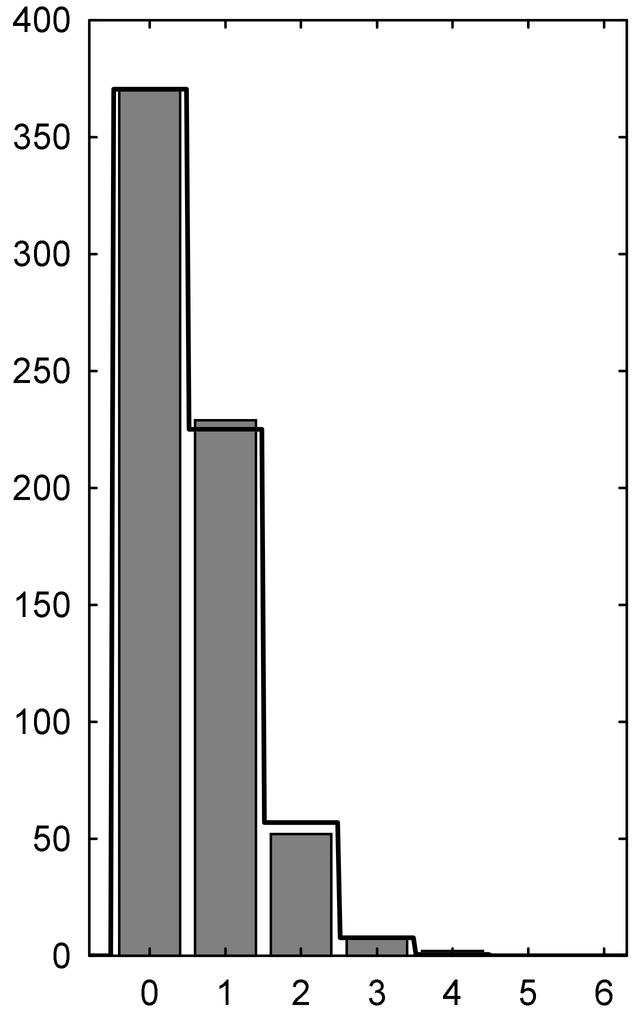
A binomial $B(6, \pi)$ distribution may be appropriate to model the marginal process distribution.

Example: time series collected on November 29th, 2005, between 10 a.m. and 9 p.m., time series of length 661.

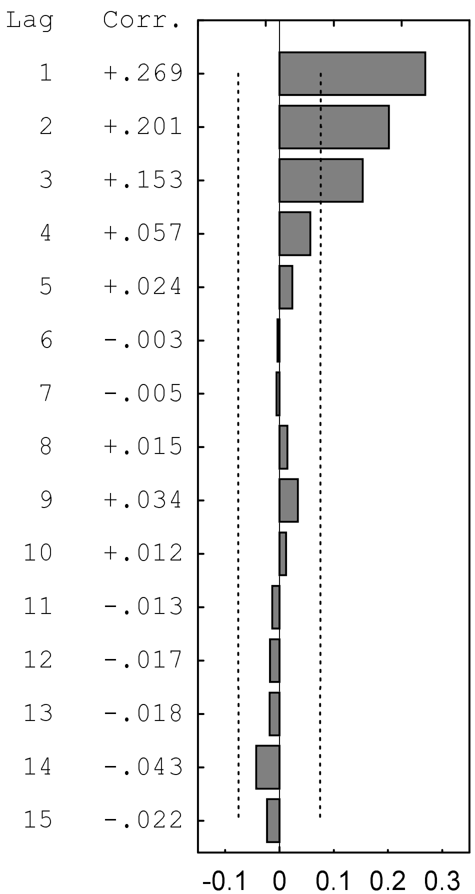


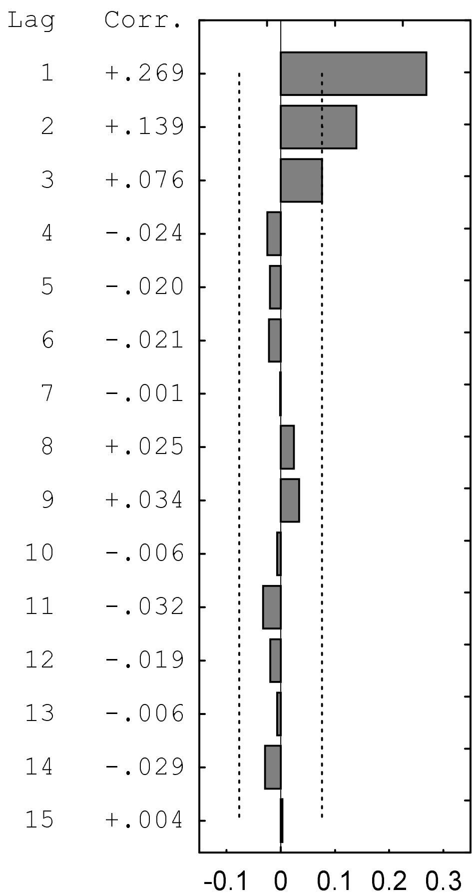
Example: Access Counts

Histogram of the data with $\widehat{\pi} := \bar{X}_T/6 = 0.09203$, i. e., binomial $B(6, 0.092)$ distribution.



Autocorrelation and partial autocorrelation function:





Autocorrelation structure is similar to that of a usual AR(p) model, with model order $p \leq 3$.

$\Rightarrow$ Try to fit binomial AR(p) – Independent Thinnings model of order $p \leq 3$.

Example: Access Counts

p	Method	$\widehat{\pi}$	$\widehat{\rho}$	$\widehat{\phi}_1$	$\widehat{\phi}_2$	AIC	BIC
1	YW	0.09203	0.2685				
	CLS	0.09217	0.2687				
	ML	0.09245	0.2598			1232	1241
2	YW	0.09203	0.3705	0.6237			
	CLS	0.09237	0.3701	0.6226			
	ML	0.09261	0.3406	0.6372		1224	1238
3	YW	0.09203	0.4186	0.5266	0.2910		
	CLS	0.09254	0.4175	0.5254	0.2904		
	ML	0.09318	0.3912	0.5229	0.2793	1223	1241

Conclusion

- A new family of autoregressive models for time series of binomial counts.

- Analyzed autocorrelation structure in general and of two special cases.

- Promising from practical point of view: Binomial AR(p) – Independent Thinnings model, close to standard AR(p) models. Usefulness of this model demonstrated by example of access counts.

Thank You for Your Interest!

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