The Marginal Distribution of Compound Poisson INAR(1) Processes

HELMUT SCHMIDT UNIVERSITÄT
Universität der Bundeswehr Hamburg

Christian H. Weiß

Department of Mathematics & Statistics

Helmut Schmidt University, Hamburg

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Pedro Puig

Department of Mathematics

Universitat Autònoma de Barcelona

Helmut Schmidt Universität
Universität der Bundeswehr Hamburg

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(Poisson) INAR(1) Processes

Definition & Properties

Basic approach for **real-valued** processes:

stationary **ARMA(p,q) model**.

Let innovations $(\epsilon_t)_{\mathbb{Z}}$ be white noise, then

$$X_t \;=\; \alpha_1 \cdot X_{t-1} + \ldots + \alpha_{\mathsf{p}} \cdot X_{t-\mathsf{p}} + \epsilon_t + \ldots + \beta_{\mathsf{q}} \cdot \epsilon_{t-\mathsf{q}},$$

where $\alpha_1, \ldots, \alpha_{\mathsf{p}}, \beta_1, \ldots, \beta_{\mathsf{q}} \in \mathbb{R}$ suitably chosen.

Enumerous extension („ARMA alphabet soup“, Holan et al. (2010)) $\rightarrow$ SARIMA, ARFIMA, VARMA, GARCH, ...

Not applicable to count data processes: generally, $\alpha \cdot X \notin \mathbb{N}_0$.

Several approaches of how to avoid "multiplication problem".

Popular: Use appropriate thinning operations. (Weiß, 2008)

In this talk, we exclusively consider models based on

binomial thinning operator (Steutel & van Harn, 1979):

$$\alpha \circ X := \sum_{i=1}^{X} Y_i, \qquad \text{where } Y_i \text{ are i.i.d. Bin}(1,\alpha),$$

i. e., $\alpha \circ X \ \sim \ \text{Bin}(X,\alpha)$ and has range $\{0,\ldots,X\}$.

In particular, $E[\alpha \circ X] \ = \ E[\alpha \cdot X]$.

($\approx$ number of "survivors" from population of size X)

INAR(1) Processes

Let $(\epsilon_t)_{\mathbb{Z}}$ be i.i.d. with range $\mathbb{N}_0 = \{0, 1, \ldots\}$, denote $E[\epsilon_t] = \mu_\epsilon$, $V[\epsilon_t] = \sigma_\epsilon^2$. Let $\alpha \in (0; 1)$.

$(X_t)_{\mathbb{Z}}$ referred to as **INAR(1) process** if

$$\underbrace{X_t}_{\text{Population at time } t} = \underbrace{\alpha \circ X_{t-1}}_{\text{Survivors of time } t-1} + \underbrace{\epsilon_t}_{\text{Immigration}} .$$

plus appropr. independence assumptions. (McKenzie, 1985)

Properties:

Homogeneous Markov chain with

$$\mathbb{P}(X_t = k \mid X_{t-1} = l) = \sum_{j=0}^{\min\{k,l\}} \binom{l}{j} \alpha^j (1-\alpha)^{l-j} \cdot \mathbb{P}(\epsilon_t = k - j).$$

INAR(1) Processes

If INAR(1) process stationary (see below), then

$$\mathrm{pgf}_X(z) \;=\; \mathrm{pgf}_X(1-\alpha+\alpha z)\cdot \mathrm{pgf}_\epsilon(z).$$

In particular, if $\mu_\epsilon, \sigma_\epsilon < \infty$, we have

$$\mu_X \;=\; \frac{\mu_\epsilon}{1-\alpha}, \qquad \sigma_X^2 \;=\; \mu_X\cdot\frac{\frac{\sigma_\epsilon^2}{\mu_\epsilon}+\alpha}{1+\alpha}.$$

(Note: X_t equidispersed iff ϵ_t equidispersed.)

Autocorrelation function: $\rho_X(k) = \alpha^k$, i. e., AR(1)-type.

For further properties and references, see Weiß (2008).

Poisson INAR(1) Processes

Most popular instance of INAR(1) family:

Poisson INAR(1) model, $X_t \ = \ \alpha \circ X_{t-1} \ + \ \epsilon_t$.

Here, innovations $(\epsilon_t)_{\mathbb{Z}}$ i.i.d. Poi(λ), such that $\mu_\epsilon = \sigma_\epsilon^2 = \lambda$.

Stationary marginal distribution:

also Poisson distribution, Poi($\frac{\lambda}{1-\alpha}$), because:

additivity of Poisson distribution, and

Poisson distribution **invariant w.r.t. binomial thinning**:

If $X \sim$ Poi(μ), then $\alpha \circ X \sim$ Poi($\alpha \cdot \mu$).

Poisson INAR(1) has **equidispersed** marginals.

Modifications to get **overdispersed** marginals:

- modify thinning operation (Weiß, 2008), or
- modify distribution of innovations, e. g., negative binomially (NB) distributed ϵ_t's.

In the following, we more generally consider **compound Poisson** (CP) distributed innovations.

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Compound Poisson INAR(1) Processes

Definition & Properties

Compound Poisson Distribution

$Y_1, Y_2, \ldots$ i.i.d. with range $\mathbb{N} = \{1, 2, \ldots\}$,
denote pgf as $H(z)$ (**compounding distribution**).

Let $N \sim \text{Poi}(\lambda)$, independent of $Y_1, Y_2, \ldots$

$\epsilon := Y_1 + \ldots + Y_N$ **compound Poisson distributed**:
$\epsilon \sim \text{CP}(\lambda, H)$.

Then $\text{pgf}_\epsilon(z) = \exp\big(\lambda(H(z) - 1)\big)$.

More precisely:
$\epsilon \sim \text{CP}_\nu(\lambda, H)$ if $H(z) = h_1 z + \ldots + h_\nu z^\nu$ with $h_\nu > 0$,
$\epsilon \sim \text{CP}_\infty(\lambda, H)$ if Y_i have infinite range.

Properties:

- ϵ equidispersed iff $\nu = 1$: $\quad \text{CP}_1(\lambda, H) = \text{Poi}(\lambda)$.

  ϵ overdispersed iff $\nu > 1$, e. g., ...

- $\text{CP}_\nu(\lambda, H)$ with $\nu < \infty$: **Hermite distribution** of order ν.

- $\text{CP}_\nu(\lambda, H)$ with $\nu < \infty$ and $h_x = 1/\nu$ for $x = 1, \ldots, \nu$:

  Poisson distribution of order ν, abbr. $\text{Poi}_\nu(\lambda)$.

- $\text{NB}(n, \pi)$-distribution: $\quad \text{CP}_\infty(\lambda, H)$ with

  $\lambda \;:=\; -n \ln \pi$ and $H(z)$ being pgf of log-series distribution:

$$H(z) \;=\; \frac{\ln\big(1-(1-\pi)z\big)}{\ln \pi} \;=\; \sum_{k=1}^{\infty} \frac{(1-\pi)^k}{-k \ln \pi}\, z^k.$$

Properties: (cont.)

- **Additivity:**

  If X_1, X_2 independent with $X_i \sim \text{CP}_\nu(\lambda_i, H_i)$ (common ν), then $X_1 + X_2$ $\text{CP}_\nu(\lambda, H)$-distributed.

- **Invariance w.r.t. binomial thinning**:

  If $X \sim \text{CP}_\nu(\lambda, H)$, then $\alpha \circ X \sim \text{CP}_\nu(\eta, G)$.

- Count data model parametrized by ν first factorial cumulants closed under addition and under binomial thinning *iff* it has CP_ν-distribution.

  (Puig & Valero, 2007; Schweer & Weiß, 2014)

INAR(1) process $(X_t)_{t\in\mathbb{Z}}$ referred to as

CPINAR(1) process

if innovations $(\epsilon_t)_{\mathbb{Z}}$ i.i.d. $\text{CP}_\nu(\lambda, H)$ (possibly $\nu = \infty$).

$\nu = 1$: Poisson INAR(1) model.

Popular for $\nu = \infty$: NB-innovations, NB-INAR(1) model.

Heathcote (1966), Schweer & Weiß (2014):

If $H'(1) < \infty$, then unique **stationary** marginal distribution of $(X_t)_{t\in\mathbb{Z}}$ is $\text{CP}_\nu(\eta, G)$, where

$$\eta\big(G(z) - 1\big) \;=\; \lambda \sum_{i=0}^{\infty} \big(H(1 - \alpha^i + \alpha^i z) - 1\big).$$

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Compound Poisson INAR(1) Processes

Marginal Distribution

Compound Poisson INAR(1): Marginal Distribution

See above:

If INAR(1)'s innovations $(\epsilon_t)_{\mathbb{Z}}$ CP$_\nu$-distributed (possibly $\nu = \infty$), then INAR(1)'s observations $(X_t)_{t\in\mathbb{Z}}$ also CP$_\nu$-distributed, with same compounding order ν.

E.g., for the purpose of likelihood computations:

How to compute **efficiently and precisely** pmf of **marginal distribution** of $(X_t)_{t\in\mathbb{Z}}$?

If compounding order **finite**, $\nu < \infty$, then marginal distribution can be computed exactly!

Compound Poisson INAR(1): Marginal Distribution

Scheme: (see Weiß & Puig (2015) for details & references)

1. Compute parameters η and $g_1, \ldots, g_\nu$ of marginal distribution:

   (a) If innovations' param. λ and $h_1, \ldots, h_\nu$ readily available, solve following linear equations in η and $g_1, \ldots, g_\nu$:

$$g_1 + \ldots + g_\nu = 1, \qquad \frac{\lambda}{\eta} - (1-\alpha)\cdot g_1 - \ldots - (1-\alpha)^\nu \cdot g_\nu = 0,$$

$$h_k \cdot \frac{\lambda}{\eta} - (1-\alpha^k)\cdot g_k + \alpha^k \sum_{i=k+1}^{\nu} \binom{i}{k} (1-\alpha)^{i-k} \cdot g_i = 0, \quad k = \nu, \ldots, 2.$$

   (b) If innovations' CP$_\nu$-distr. via fact. cumul. $\kappa_{(1),\epsilon}, \ldots, \kappa_{(\nu),\epsilon}$, then first $\kappa_{(1),X}, \ldots, \kappa_{(\nu),X}$ via $\kappa_{(n),X} = \frac{\kappa_{(n),\epsilon}}{1-\alpha^n}$, afterwards expand $\eta\big(G(z)-1\big) := \sum_{r=1}^{\nu} \frac{\kappa_{(r),X}}{r!}\cdot(z-1)^r$.

Scheme: (see Weiß & Puig (2015) for details & references)

1. (. . .)
2. Recursive scheme for computation of pmf $P(X=k)$:

$$P(X=0) \;=\; e^{-\eta}, \qquad \text{(“Panjer recursion”)}$$

$$P(X=k) \;=\; \frac{\eta}{k}\cdot \sum_{j=1}^{\min\{k,\nu\}} j\, g_j \cdot P(X=k-j) \quad \text{for } k\geq 1.$$

In the case $\nu=\infty$, however, Step 1 is crucial point.

Idea: Replace infinite compounding structure by finite one, then continue with above Scheme.

Hermite-ν approximation, type 1:

For innovations ϵ_t being $\text{CP}_\infty(\lambda, H)$-distributed,

define ν-th order approximation $\text{CP}_\nu(\tilde{\lambda}, \tilde{H})$ by

$$\tilde{\lambda} := \lambda, \qquad \tilde{h}_1 := h_1,\ \ldots,\ \tilde{h}_{\nu-1} := h_{\nu-1}, \quad \tilde{h}_\nu := 1 - \textstyle\sum_{k=1}^{\nu-1} h_k.$$

Then proceed according to Steps 1(a) and 2 of above Scheme.

Preserves $P(\epsilon = 0), \ldots, P(\epsilon = \nu - 1)$,

approximates remaining innovations' probabilities.

1st order approximation = Poisson approximation.

Hermite-ν approximation, type 2:

For innovations ϵ_t being $\text{CP}_\infty(\lambda, H)$-distributed with mean μ_ϵ, define ν-th order approximation $\text{CP}_\nu(\tilde{\lambda}, \tilde{H})$ by

$$\tilde{h}_1 := \frac{h_1}{\sum_{k=1}^{\nu} h_k}, \quad \ldots, \quad \tilde{h}_\nu := \frac{h_\nu}{\sum_{k=1}^{\nu} h_k}, \qquad \tilde{\lambda} := \frac{\mu_\epsilon}{\sum_{k=1}^{\nu} k\,\tilde{h}_k}.$$

Then proceed according to Steps 1(a) and 2 of above Scheme.

Preserves observations' mean μ_X,

approximates all innovations' probabilities.

1st order approximation = Poisson approximation.

Performance evaluation:

NB-INAR(1) model with NB(n, π)-distributed innovations,

different levels of mean parameter n,

dispersion parameter π, autocorrelation parameter α.

Benchmark:

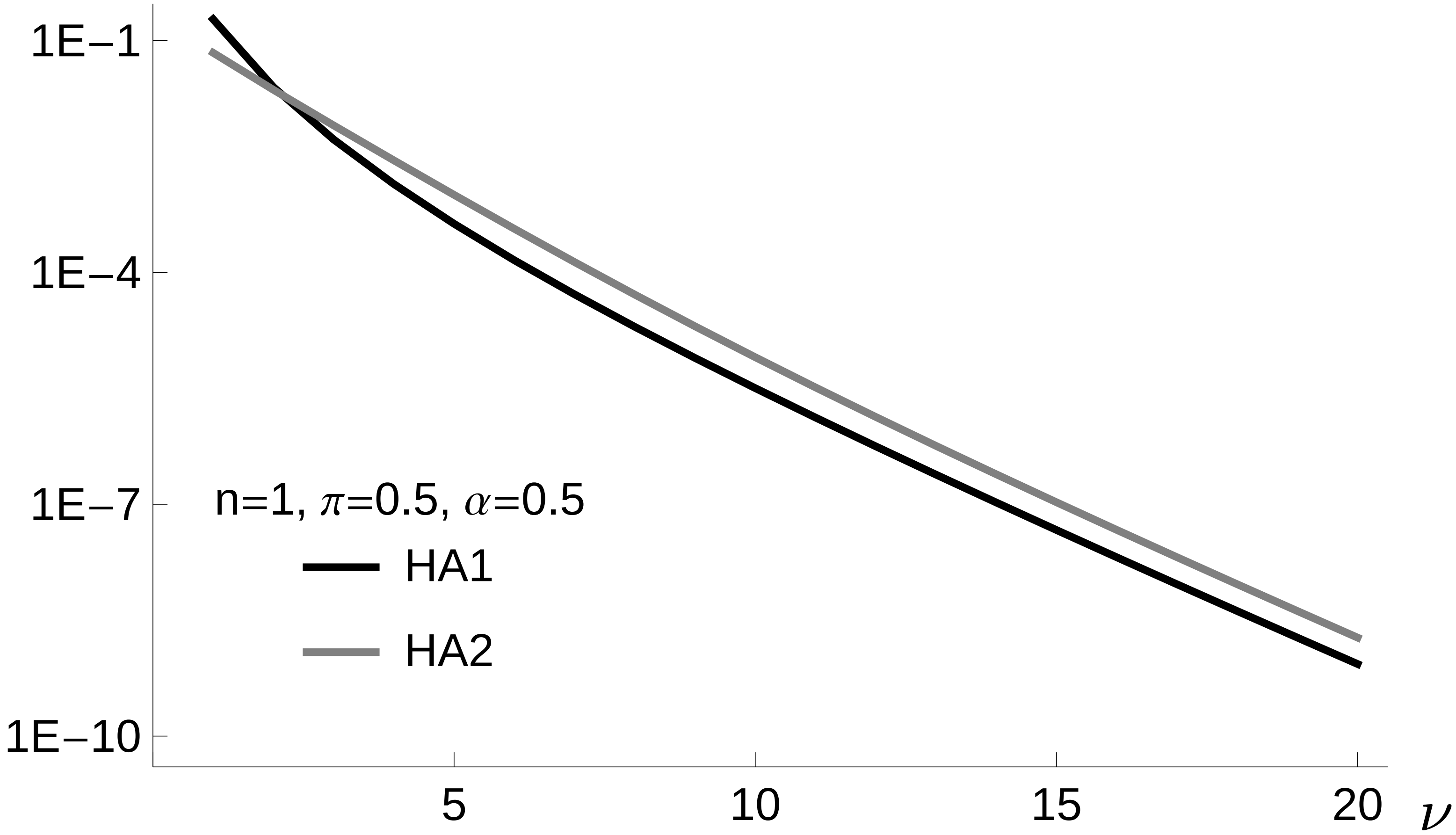
Numerically exact Markov chain approach (Weiß, 2010).

Concerning computational **efficiency:**

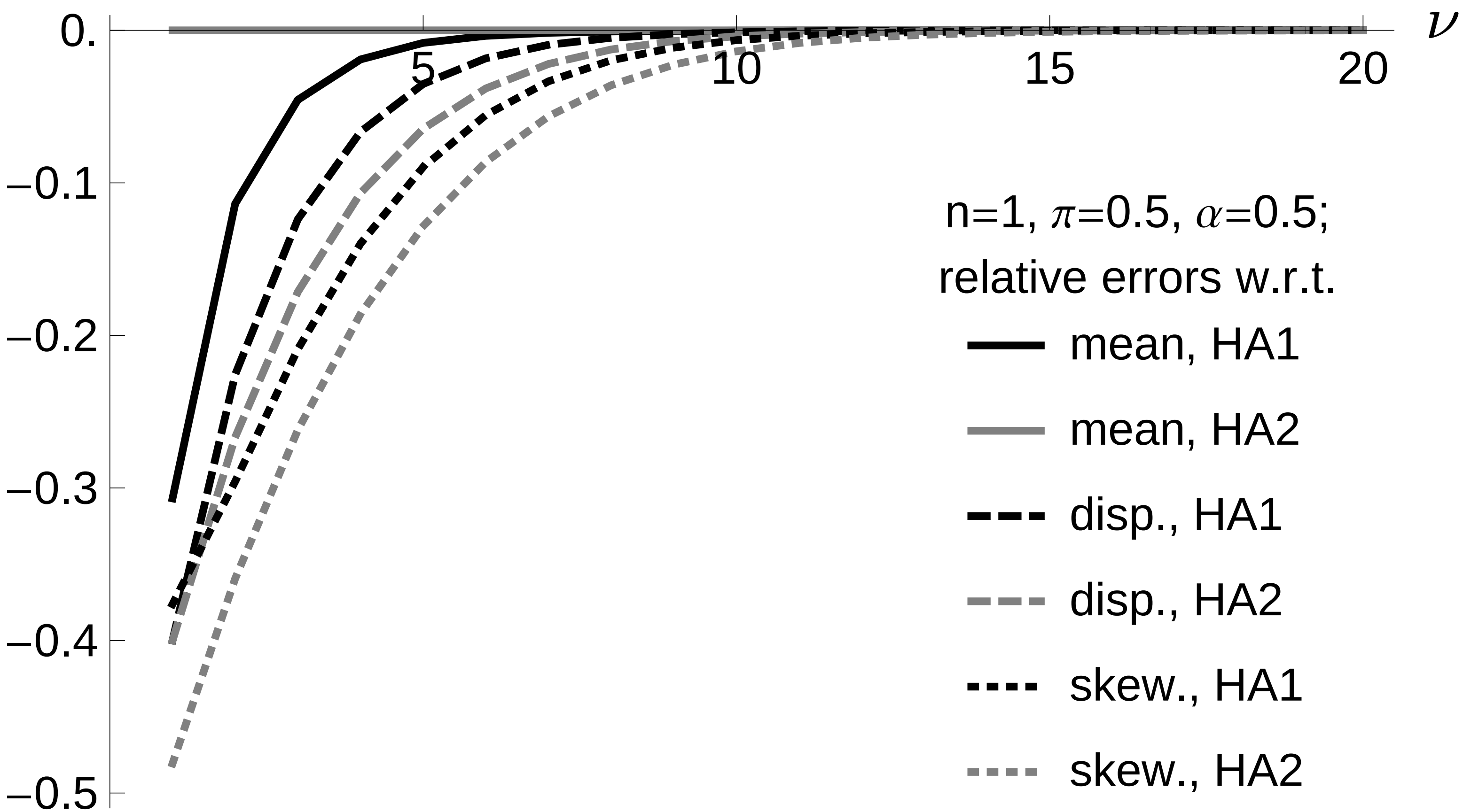
Computing time for approximations: $< 0.1\,$s.

Computing time for MC approach: $70 - 100\,$s.

Overall quality of approximation via Kullback-Leibler divergence:



Relative errors $\dfrac{\text{approx} - \text{true}}{\text{true}}$ for mean, dispersion index and skewness:



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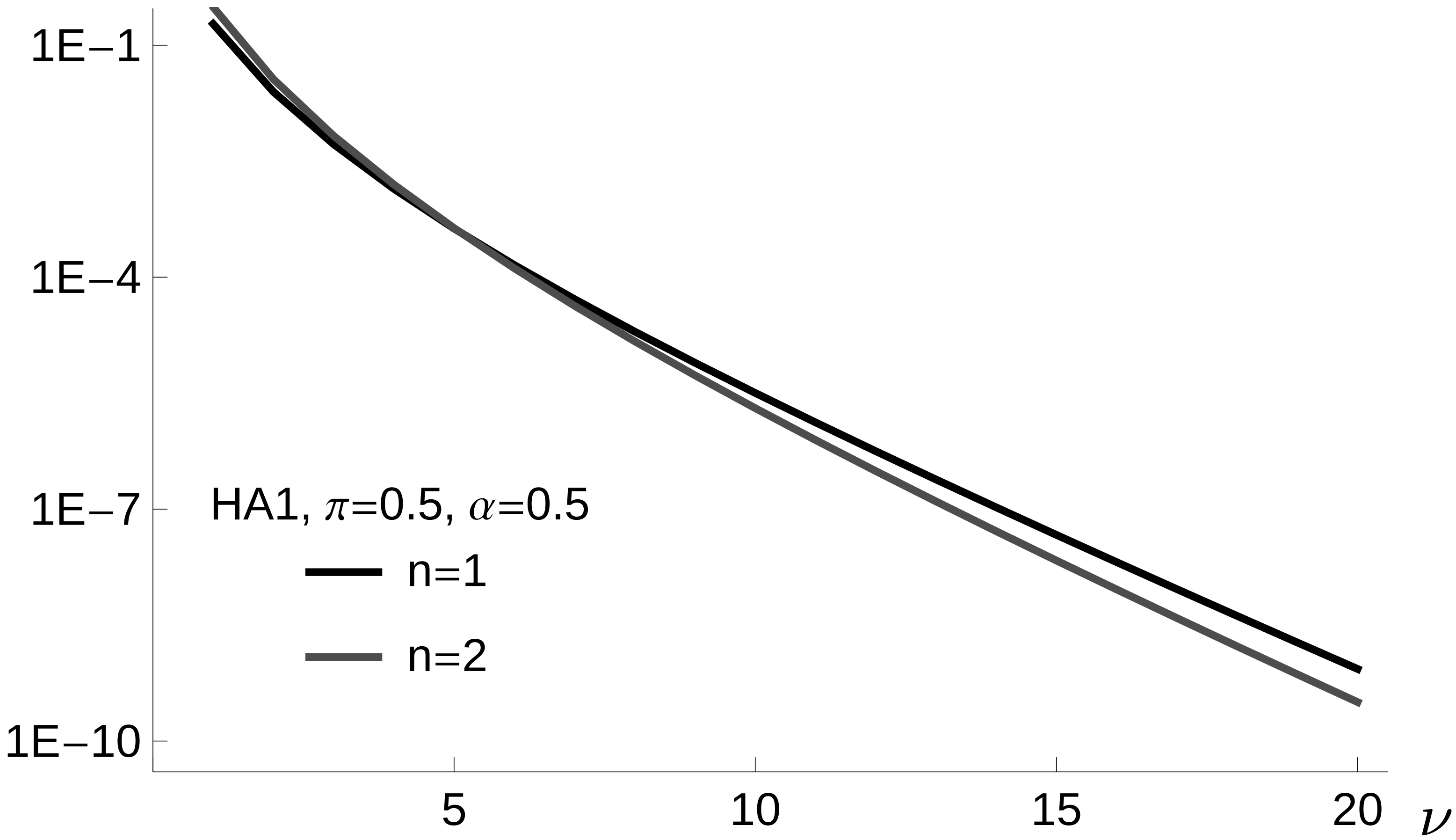
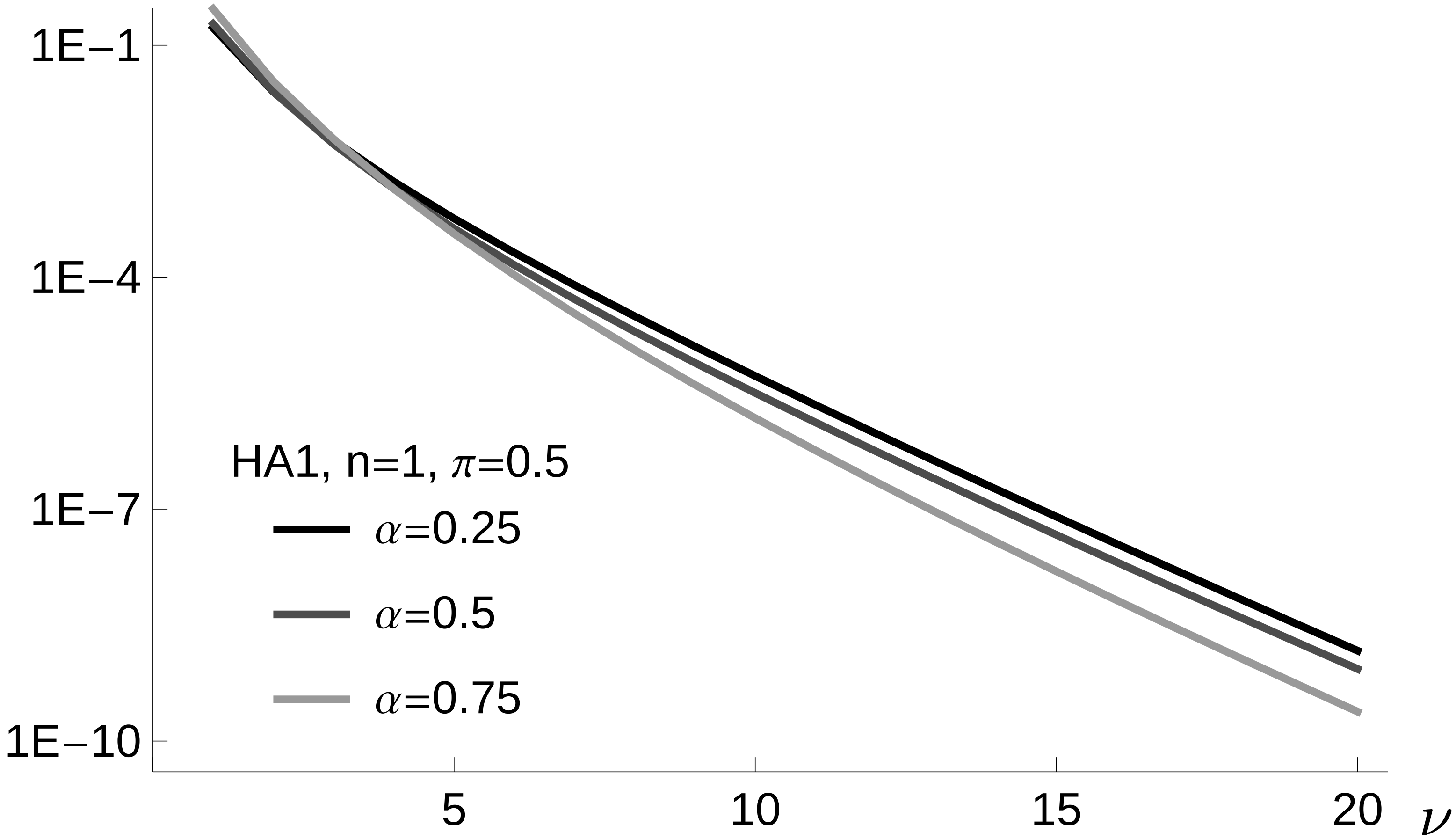
Overall quality of type 1 approximation via KL divergence:



Compound Poisson INAR(1): Marginal Distribution

Overall quality of type 1 approximation via KL divergence:



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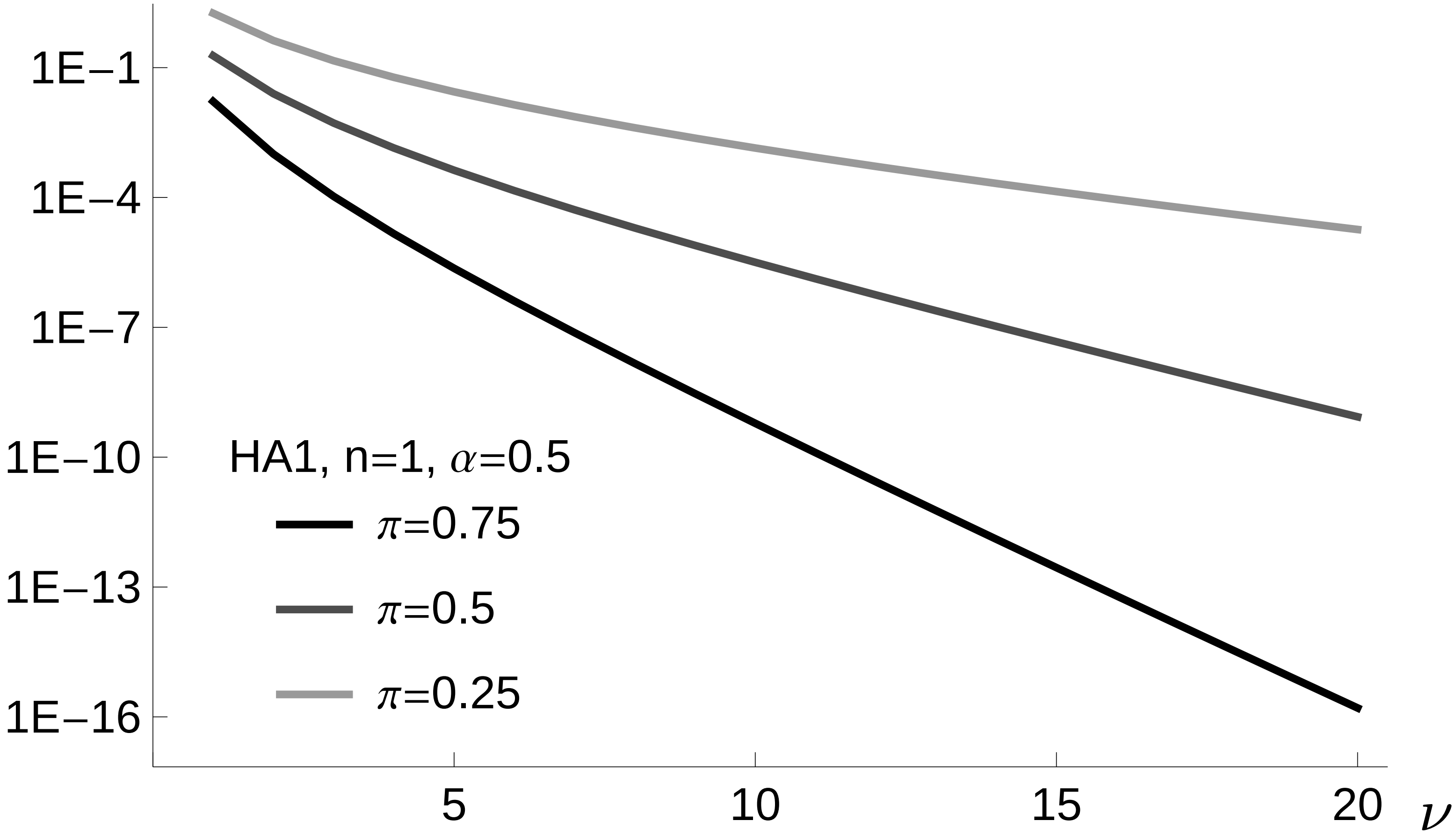

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Conclusions

. . . and Future Research

Conclusions

- Type 1 approximation performs best concerning KL divergence, and concerning dispersion index or skewness.
- If one chooses $\nu \geq 15$, then difference between exact distribution and type 1 approximation negligible.
- Both approximations much faster than MC approach.

Future research:

empirical version of the Hermite approximation, e. g., based on first ν empirical factorial cumulants.
($\rightarrow$ semi-parametric estimation of INAR(1)'s marginal)

Thank You

for Your Interest!

HELMUT SCHMIDT UNIVERSITÄT
Universität der Bundeswehr Hamburg

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Christian H. Weiß

Department of Mathematics & Statistics

Helmut Schmidt University, Hamburg

weissc@hsu-hh.de

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