Compound Poisson INAR(1) Processes: Stochastic Properties and Testing for Overdispersion

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INAR(1) Model for Time Series of Counts

Motivation & Properties

Popular for **real-valued** stationary processes:

ARMA(p,q) model. Let $(\epsilon_t)_{\mathbb{Z}}$ white noise, then

$$X_t \;=\; \alpha_1\cdot X_{t-1}+\ldots+\alpha_{\mathsf{p}}\cdot X_{t-\mathsf{p}}+\epsilon_t+\beta_1\cdot\epsilon_{t-1}+\ldots+\beta_{\mathsf{q}}\cdot\epsilon_{t-\mathsf{q}},$$

where $\alpha_1,\ldots,\alpha_{\mathsf{p}},\beta_1,\ldots,\beta_{\mathsf{q}}\in\mathbb{R}$ suitably chosen.

Autocorrelation via **Yule-Walker equations**:

$$\rho_X(k) \;=\; \sum_{j=1}^{\mathsf{p}} \alpha_j\cdot\rho_X(|k-j|) + \frac{\sigma_\epsilon^2}{\sigma_X^2}\cdot\sum_{i=0}^{\mathsf{q}-k} \beta_{i+k}\cdot a_i + \delta_{k0}\cdot\frac{\sigma_\epsilon^2}{\sigma_X^2}.$$

Example: AR(1) model $X_t \;=\; \alpha\cdot X_{t-1}+\epsilon_t$ with $\rho_X(k)=\alpha^k$.

Not applicable to count data processes: generally, $\alpha\cdot X\notin\mathbb{N}_0$.

Several approaches in literature of

how to avoid the "multiplication problem".

In this talk, we consider models based on

binomial thinning operator (Steutel & van Harn, 1979):

$$\alpha \circ X := \sum_{i=1}^{X} Y_i, \qquad \text{where } Y_i \text{ are i.i.d. Bin}(1, \alpha),$$

i. e., $\alpha \circ X \ \sim\ \text{Bin}(X, \alpha)$ and has range $\{0, \ldots, X\}$.

($\approx$ number of "survivors" from population of size X)

Let $(\epsilon_t)_{\mathbb{Z}}$ be i.i.d. with range $\mathbb{N}_0 = \{0, 1, \ldots\}$, denote $\mathbb{E}[\epsilon_t] = \mu_\epsilon$, $\mathrm{Var}[\epsilon_t] = \sigma_\epsilon^2$. Let $\alpha \in (0; 1)$.

$(X_t)_{\mathbb{Z}}$ referred to as **INAR(1) process** if

$$X_t \;=\; \alpha \circ X_{t-1} \;+\; \epsilon_t \qquad \text{for } t \geq 1,$$

together with appropriate independence assumptions.

(McKenzie, 1985)

Homogeneous Markov chain with

$$\mathbb{P}(X_t = k \mid X_{t-1} = l) \;=\; \sum_{j=0}^{\min\{k,l\}} \binom{l}{j} \alpha^j (1-\alpha)^{l-j} \cdot \mathbb{P}(\epsilon_t = k-j).$$

If INAR(1) process stationary (see below), then

$$\text{pgf}_X(z) \;=\; \text{pgf}_X(1-\alpha+\alpha z)\cdot \text{pgf}_\epsilon(z).$$

In particular, if $\mu_\epsilon, \sigma_\epsilon < \infty$, we have

$$\mu_X \;=\; \frac{\mu_\epsilon}{1-\alpha}, \qquad \sigma_X^2 \;=\; \mu_X \cdot \frac{\frac{\sigma_\epsilon^2}{\mu_\epsilon}+\alpha}{1+\alpha}.$$

(Note: X_t equidispersed iff ϵ_t equidispersed.)

Autocorrelation function: $\rho_X(k) = \alpha^k$, i. e., AR(1)-type.

For further properties and references, see Weiß (2008).

Most popular instance of INAR(1) family:

Poisson INAR(1) model, $$X_t \;=\; \alpha \circ X_{t-1} \;+\; \epsilon_t.$$

Here, innovations $(\epsilon_t)_{\mathbb{Z}}$ i.i.d. Poi(λ), such that $\mu_\epsilon = \sigma_\epsilon^2 = \lambda$.

Stationary marginal distribution:

also Poisson distribution, Poi($\frac{\lambda}{1-\alpha}$), because:

additivity of Poisson distribution, and

Poisson distribution invariant to binomial thinning:

If $X \sim$ Poi(μ), then $\alpha \circ X \sim$ Poi($\alpha \cdot \mu$).

Poisson INAR(1) has **equidispersed** marginals.

Modifications to get **overdispersed** marginals:

- modify thinning operation (Weiß, 2008), or
- modify distribution of innovations, e. g.,
  as in Jung et al. (2005), Pedeli & Karlis (2011):
  negative binomially (NB) distributed ϵ_t's.

In the following, we more generally consider
compound Poisson (CP) distributed innovations.

Compound Poisson INAR(1) Processes

Definition & Properties

Compound Poisson Distribution

$Y_1, Y_2, \ldots$ i.i.d. with range $\mathbb{N} = \{1, 2, \ldots\}$,
denote pgf as $H(z)$.

Let $N \sim$ Poi(λ), independent of $Y_1, Y_2, \ldots$

$\epsilon := Y_1 + \ldots + Y_N$ **compound Poisson distributed**:
$\epsilon \sim$ CP(λ, H).

Then $\quad \text{pgf}_\epsilon(z) \ = \ \exp\big(\lambda(H(z) - 1)\big)$. (Feller, 1968)

More precisely:
$\epsilon \sim \text{CP}_\nu(\lambda, H)$ if $H(z) = h_1 z + \ldots + h_\nu z^\nu$ with $h_\nu > 0$,
$\epsilon \sim \text{CP}_\infty(\lambda, H)$ if Y_i have infinite range.

Some examples:

- $\mathsf{CP}_1(\lambda, H) = \mathsf{Poi}(\lambda)$.
- CP_ν with $\nu < \infty$ and $h_x = 1/\nu$ for $x = 1, \ldots, \nu$:
  Poisson distribution of order ν, abbr. $\mathsf{Poi}_\nu(\lambda)$.
- $\mathsf{NB}(n, \pi)$-distribution: CP_∞ with $\lambda := -n \ln \pi$ and

$$H(z) = \frac{\ln\big(1-(1-\pi)z\big)}{\ln \pi} = \sum_{k=1}^{\infty} \frac{(1-\pi)^k}{-k \ln \pi}\, z^k.$$

Important properties:

- $\kappa_{\epsilon,r} \;=\; \lambda \cdot \mu_{Y,r}$ (Aki, 1985)

  $\Rightarrow$

  ϵ equidispersed iff $\nu = 1$,

  ϵ overdispersed iff $\nu > 1$.

- If X_1, X_2 independent with $X_i \sim \text{CP}_\nu(\lambda_i, H_i)$ (common ν), then $X_1 + X_2$ $\text{CP}_\nu(\lambda, H)$-distributed.

- If $X \sim \text{CP}_\nu(\lambda, H)$, then $\alpha \circ X \sim \text{CP}_\nu(\mu, G)$.

INAR(1) process $(X_t)_{t\in\mathbb{Z}}$ referred to as
CPINAR(1) process
if innovations $(\epsilon_t)_{\mathbb{Z}}$ i.i.d. $\text{CP}_\nu(\lambda, H)$ (possibly $\nu = \infty$).

$\nu = 1$: Poisson INAR(1) model.
NB-innovations: Jung et al. (2005), Pedeli & Karlis (2011).

Using Heathcote (1966), we show:

If $H'(1) < \infty$, then unique **stationary** marginal distribution of $(X_t)_{t\in\mathbb{Z}}$ is $\text{CP}_\nu(\mu, G)$, where

$$\mu\big(G(z)-1\big) \;=\; \lambda \textstyle\sum_{i=0}^{\infty} \big(H(1-\alpha^i+\alpha^i z)-1\big).$$

Preliminary summary:

Stationary CPINAR(1) process with $\mathrm{CP}_\nu(\lambda, H)$-innovations has $\mathrm{CP}_\nu(\mu, G)$-observations (same ν), i. e.,

$(X_t)_{t\in\mathbb{Z}}$ is **equi**dispersed **iff** $\nu = 1$,
$(X_t)_{t\in\mathbb{Z}}$ is **over**dispersed **iff** $\nu > 1$.

Aim: Develop test to distinguish between

- null hypothesis of equidispersion ($\nu = 1$), and
- alternative hypothesis of overdispersion ($\nu > 1$).

Index of dispersion: $I_X \;:=\; \dfrac{\sigma_X^2}{\mu_X}, \qquad \hat{I}_X \;:=\; \dfrac{S_X^2}{\bar{X}}.$

Compound Poisson INAR(1) Processes

Using Pakes (1971), we show:

If $H'(1) < \infty$, then $(X_t)_{t\in\mathbb{Z}}$ is α-**mixing**
with exponentially decreasing weights $\alpha(n)$.

$\Rightarrow$ **central limit theorem** of Ibragimov (1962) applicable,

e. g., to $\boldsymbol{Y}_t := \big(X_t - \mu_X, X_t^2 - \mu_X^2 - \sigma_X^2\big)$:

$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \boldsymbol{Y}_t \xrightarrow{\mathcal{D}} \mathrm{N}\,(\boldsymbol{0}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Sigma} = \big(\sigma_{ij}\big)$ given by

$$\sigma_{ij} \;=\; \mathbb{E}\left[Y_{0,i}Y_{0,j}\right] + \sum_{k=1}^{\infty} \big(\mathbb{E}[Y_{0,i}Y_{k,j}] + \mathbb{E}[Y_{k,i}Y_{0,j}]\big).$$

Theorem: For $0 \leq s_1 \leq \ldots \leq s_{r-1}$ and $r \in \mathbb{N}$, denote

$$\mu(s_1, \ldots, s_{r-1}) \; := \; \mathbb{E}[X_t \cdot X_{t+s_1} \cdots X_{t+s_{r-1}}].$$

Let $(X_t)_{\mathbb{Z}}$ (arbitrary) stationary INAR(1) process, with existing moments $\mu_{\epsilon,r} := \mathbb{E}[\epsilon_t^r]$ for $r \leq 4$.

Then for any $0 \leq k \leq l \leq m$,

$$\mu(k) \; = \; \sigma_X^2 \cdot \alpha^k + \mu_X^2,$$

$$\begin{aligned} \mu(k,l) \; = \; & (\bar{\mu}_{X,3} - \sigma_X^2) \cdot \alpha^{l+k} \; + \; (1+\mu_X)\sigma_X^2 \cdot \alpha^l \\ & + \; \mu_X \sigma_X^2 \cdot (\alpha^{l-k} + \alpha^k) \; + \; \mu_X^3, \end{aligned}$$

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$$\begin{aligned}\mu(k,l,m) \;=\; & \alpha^{m+l+k}\cdot\big(\bar\mu_{X,4}-3\bar\mu_{X,3}+\sigma_X^2(2-3\sigma_X^2)\big) \;+\; \mu_X^4\\ & +\ (\bar\mu_{X,3}-\sigma_X^2)\big((2+\mu_X)\cdot\alpha^{m+l}\\ & +\ (1+\mu_X)\cdot\alpha^{m+k}+\mu_X\cdot(\alpha^{m+l-2k}+\alpha^{l+k})\big)\\ & +\ (1+\mu_X)^2\sigma_X^2\cdot\alpha^{m}\\ & +\ \mu_X(1+\mu_X)\sigma_X^2\cdot(\alpha^{m-k}+\alpha^{l})\\ & +\ \mu_X^2\sigma_X^2\cdot(\alpha^{m-l}+\alpha^{l-k}+\alpha^{k})\\ & +\ \sigma_X^4\cdot(\alpha^{m-l+k}+2\alpha^{m+l-k}).\end{aligned}$$

Corollary: Defining σ_{ij} as above, we have

$$\sigma_{11} = \sigma_X^2\,\frac{1+\alpha}{1-\alpha},$$

$$\begin{aligned}\sigma_{22} &= \big(\bar{\mu}_{X,4} - (1-2\mu_X)\,\bar{\mu}_{X,3} - 2\mu_X\sigma_X^2 - \sigma_X^4\big)\,\frac{1+\alpha^2}{1-\alpha^2} \\ &\quad +(1+2\mu_X)(\bar{\mu}_{X,3} + 2\mu_X\sigma_X^2)\,\frac{1+\alpha}{1-\alpha},\end{aligned}$$

$$\begin{aligned}\sigma_{12}=\sigma_{21} &= \frac{1}{2}(\bar{\mu}_{X,3} - \sigma_X^2)\,\frac{1+\alpha^2}{1-\alpha^2} \\ &\quad +\frac{1}{2}\big(\bar{\mu}_{X,3} + \sigma_X^2(1+4\mu_X)\big)\,\frac{1+\alpha}{1-\alpha}.\end{aligned}$$

Index of dispersion: $I_X := \dfrac{\sigma_X^2}{\mu_X}, \qquad \hat{I}_X := \dfrac{S_X^2}{\bar{X}},$

where

$$\bar{X} = \tfrac{1}{T}\sum_{t=1}^{T} X_t, \quad S_X^2 = \tfrac{1}{T}\sum_{t=1}^{T}(X_t - \bar{X})^2 = \left(\tfrac{1}{T}\sum_{t=1}^{T} X_t^2\right) - \bar{X}^2.$$

Application of delta theorem to above result yields

$$\sqrt{T}\cdot(\hat{I}_X - I_X) \xrightarrow{\mathcal{D}} \mathrm{N}\left(0,\ \sigma^2\right) \quad \text{as} \quad T \longrightarrow \infty,$$

$$\sigma^2 = \frac{1+\alpha}{1-\alpha}(\mu_X - \sigma_X^2)\left(\frac{\bar{\mu}_{X,3}}{\mu_X^3} - \frac{\sigma_X^4}{\mu_X^4}\right) + \frac{1+\alpha^2}{1-\alpha^2}\left(\frac{\bar{\mu}_{X,4}}{\mu_X^2} - \frac{\bar{\mu}_{X,3}}{\mu_X^3}(\mu_X + \sigma_X^2) + \frac{\sigma_X^4}{\mu_X^3}(1-\mu_X)\right).$$

Compound Poisson INAR(1) Processes

Testing for Overdispersion

Design of **test for overdispersion**:

H_0: $(X_t)_{t\in\mathbb{Z}}$ follows *Poisson* INAR(1) model
with parameters (λ, α).

In this case, we have

$$\sqrt{T}\cdot(\hat{I}_X - 1) \xrightarrow{\mathcal{D}} \mathrm{N}\left(0,\ 2\,\frac{1+\alpha^2}{1-\alpha^2}\right) \quad \text{as} \quad T \longrightarrow \infty.$$

$\Rightarrow$ critical value

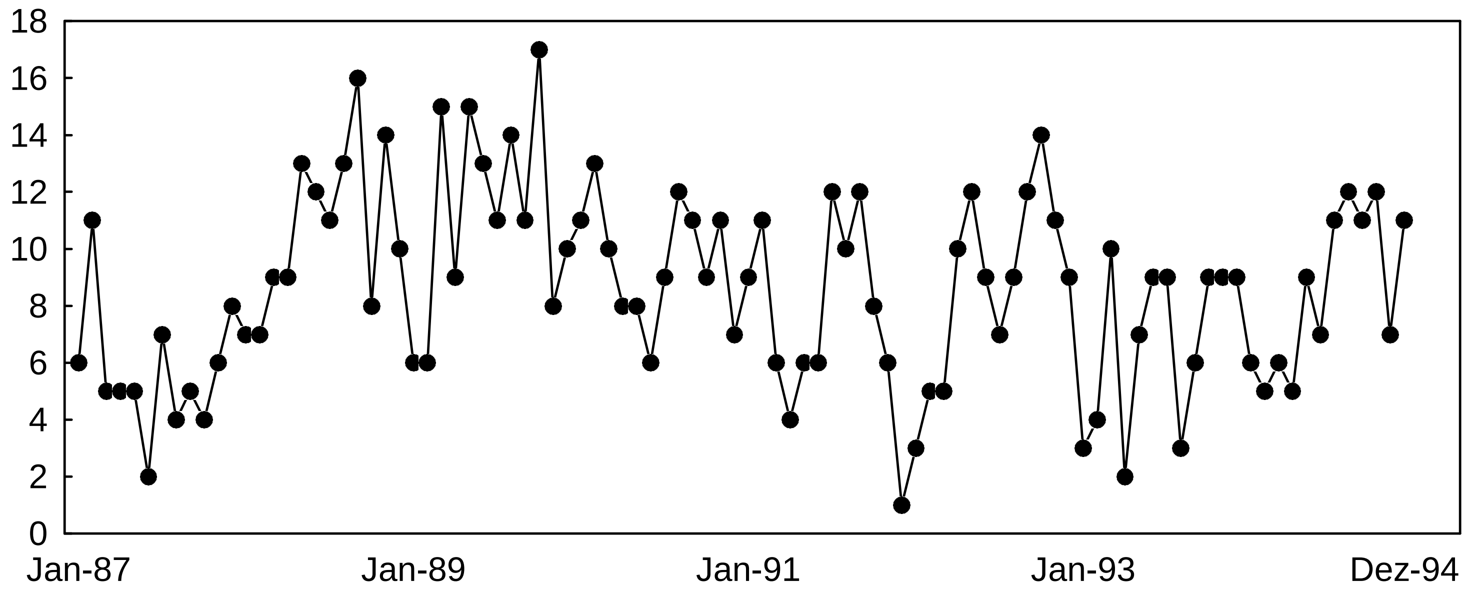
$$1 + z_{1-\beta}\cdot\sqrt{\frac{2}{T}\,\frac{1+\alpha^2}{1-\alpha^2}},$$

where $z_{1-\beta}$: $(1-\beta)$-quantile of $\mathrm{N}(0,1)$-distribution.

Example: Monthly claims counts (1987 to 1994):

burn related injuries in heavy manufacturing industry.

Source: Freeland (1998).



Example (continued): Freeland's claims counts data.

Length $T = 96$, $\bar{x} \approx 8.604$, $s_x^2 \approx 11.24$

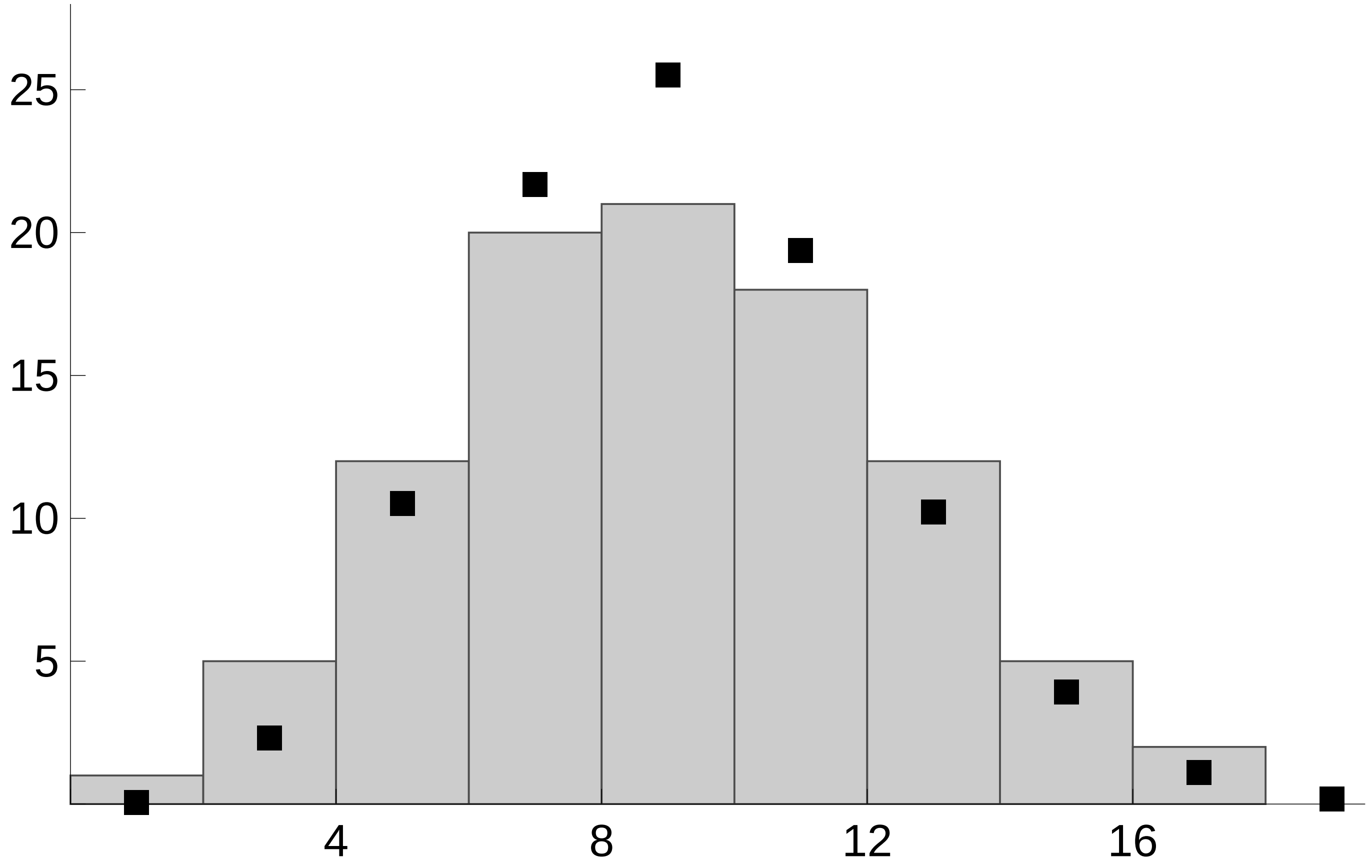
$\Rightarrow \hat{I}_y \approx 1.306$, i. e., about 31 % of empirical overdispersion.

However, plugging-in $\hat{\rho}(1) \approx 0.452$ instead of α,

critical value 1.292 (significance level $\beta = 0.05$),

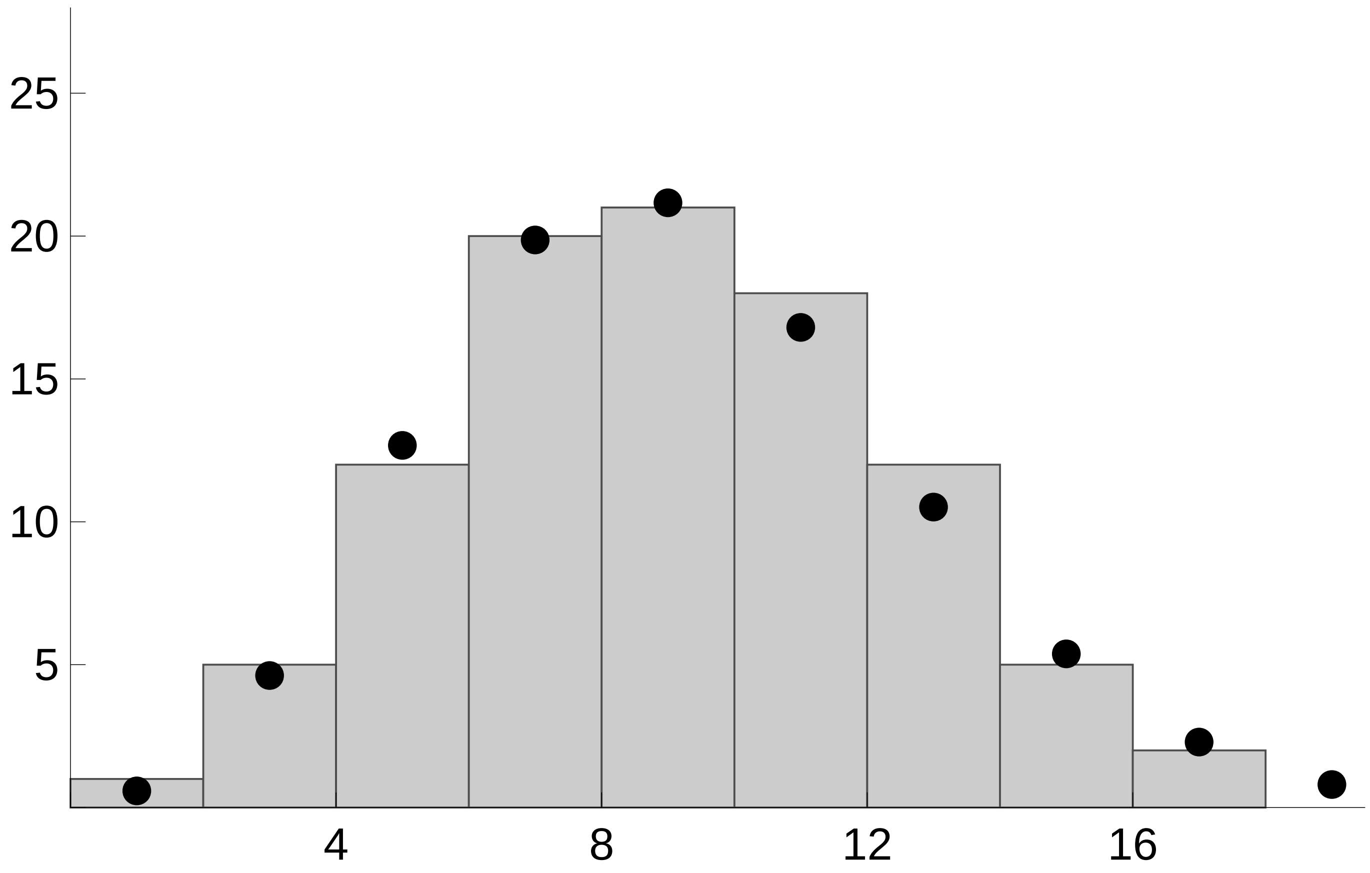
i. e., quite narrow decision.

Example (continued): Freeland's claims counts data.



Histogram with fitted Poi-marginal distribution.

Example (continued): Freeland's claims counts data.



Histogram with fitted Poi_2-marginal distribution.

Our above result

$$\sqrt{T} \cdot (\hat{I}_X - I_X) \xrightarrow{\mathcal{D}} \mathsf{N}\big(0,\ \sigma^2\big) \quad \text{as} \quad T \longrightarrow \infty,$$

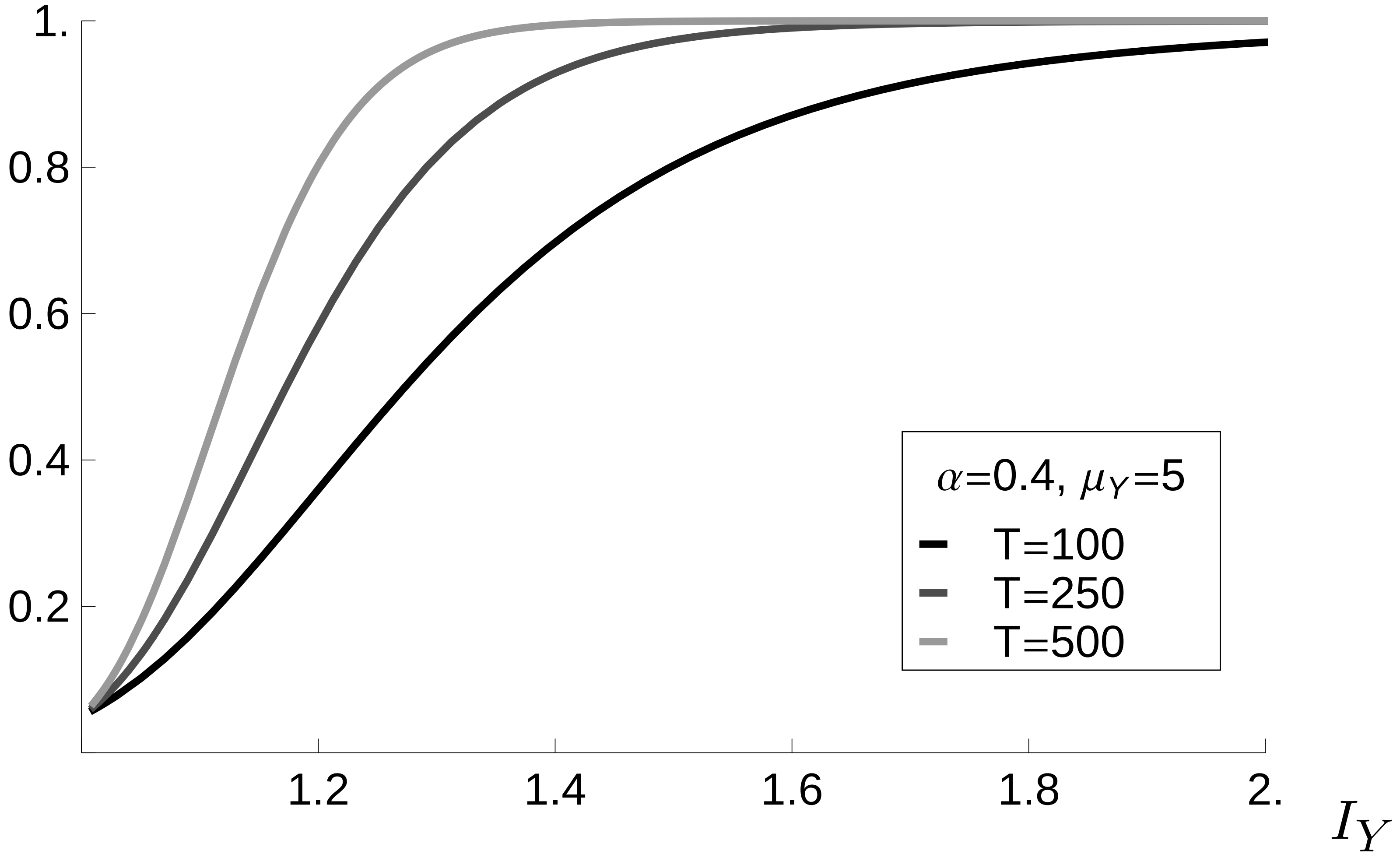
$$\begin{aligned} \sigma^2 \ = & \frac{1+\alpha}{1-\alpha}\,(\mu_X - \sigma_X^2)\left(\frac{\bar{\mu}_{X,3}}{\mu_X^3} - \frac{\sigma_X^4}{\mu_X^4}\right) \\ & + \ \frac{1+\alpha^2}{1-\alpha^2}\left(\frac{\bar{\mu}_{X,4}}{\mu_X^2} - \frac{\bar{\mu}_{X,3}}{\mu_X^3}\,(\mu_X + \sigma_X^2) + \frac{\sigma_X^4}{\mu_X^3}\,(1-\mu_X)\right), \end{aligned}$$

also applicable to overdispersed alternative!

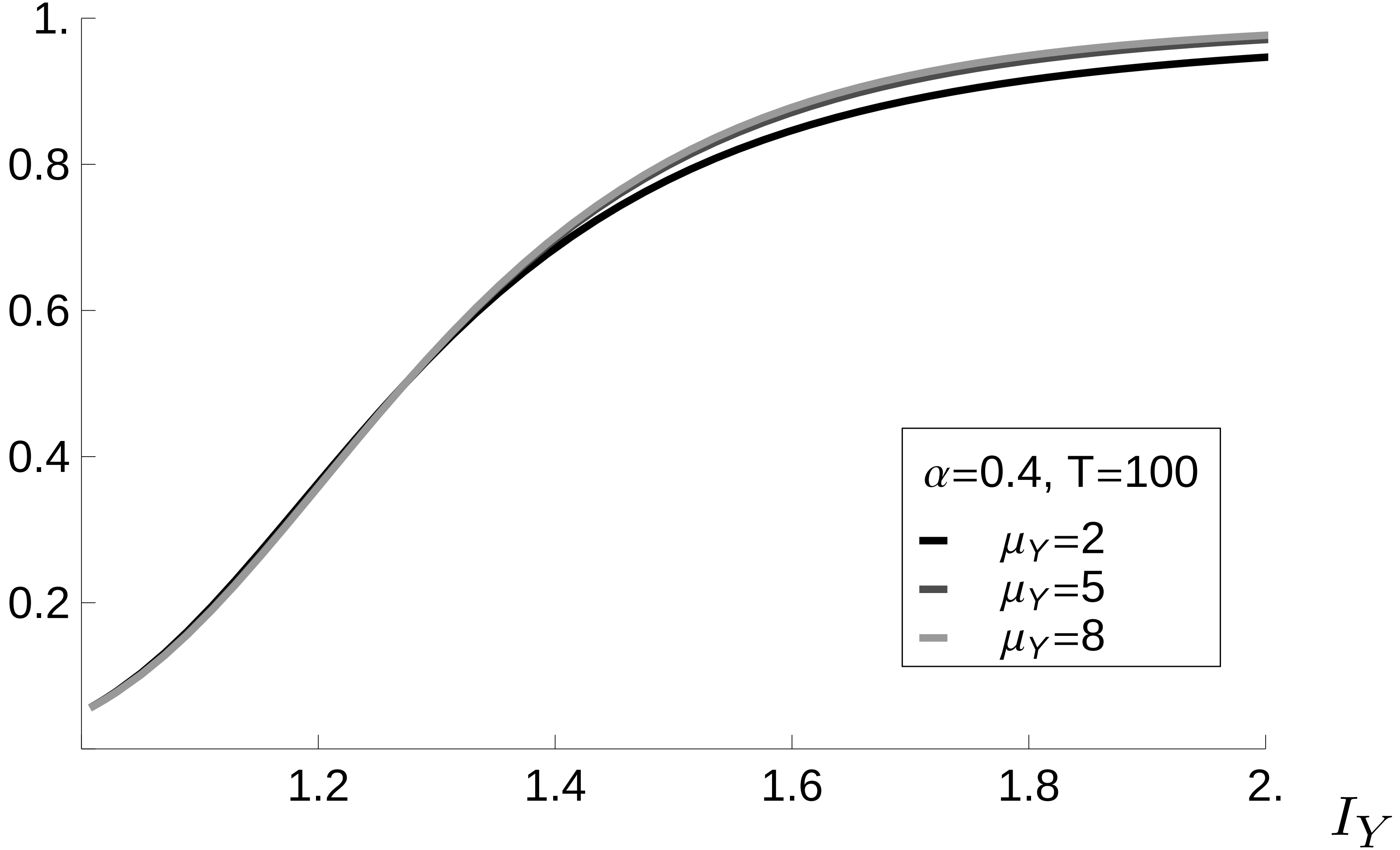
$\Rightarrow$ **Power analysis**, e. g., for NB(n, π)-innovations:

n controls mean, π overdispersion, α autocorrelation.

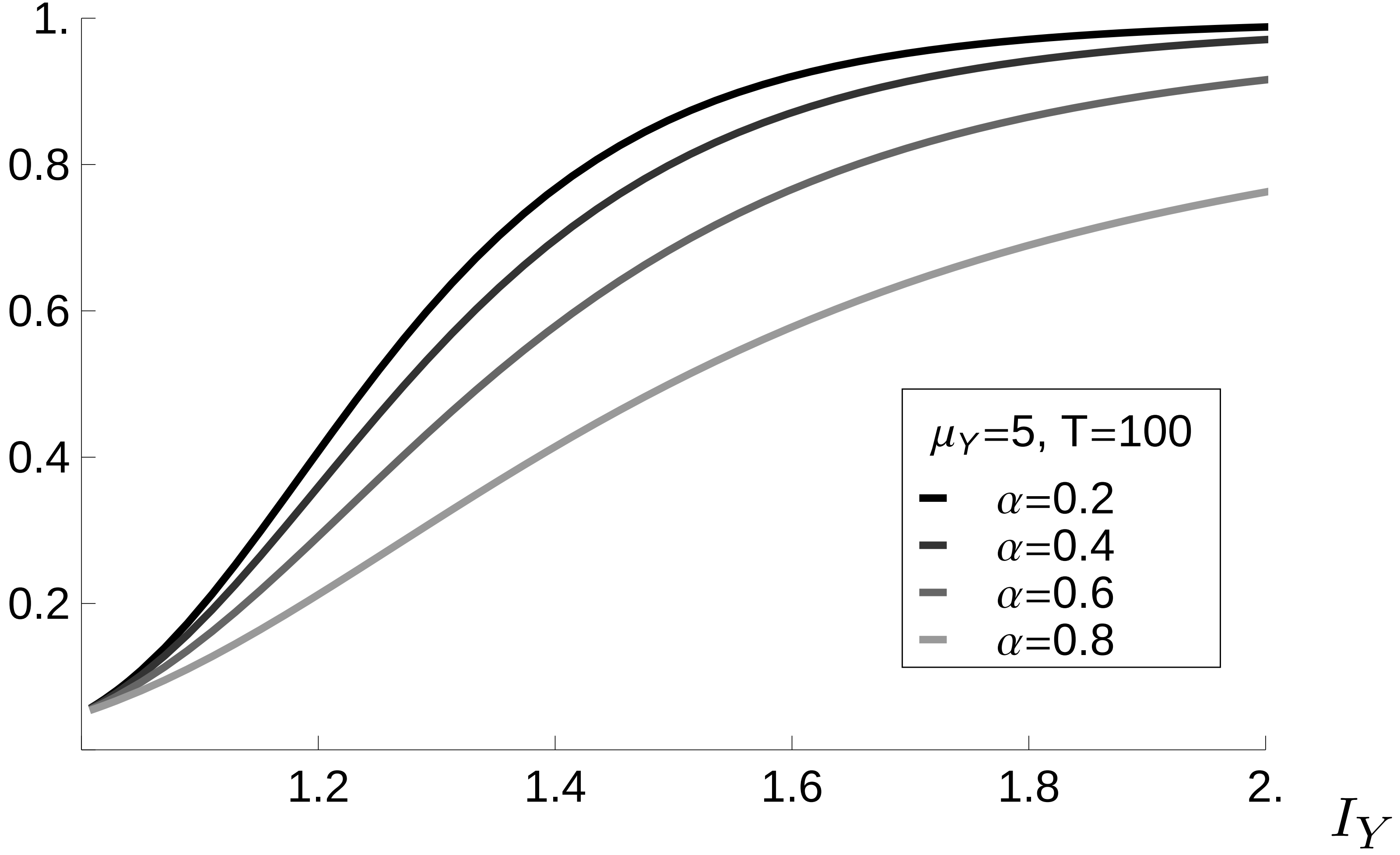
Power analysis for NB(n, π)-innovations:



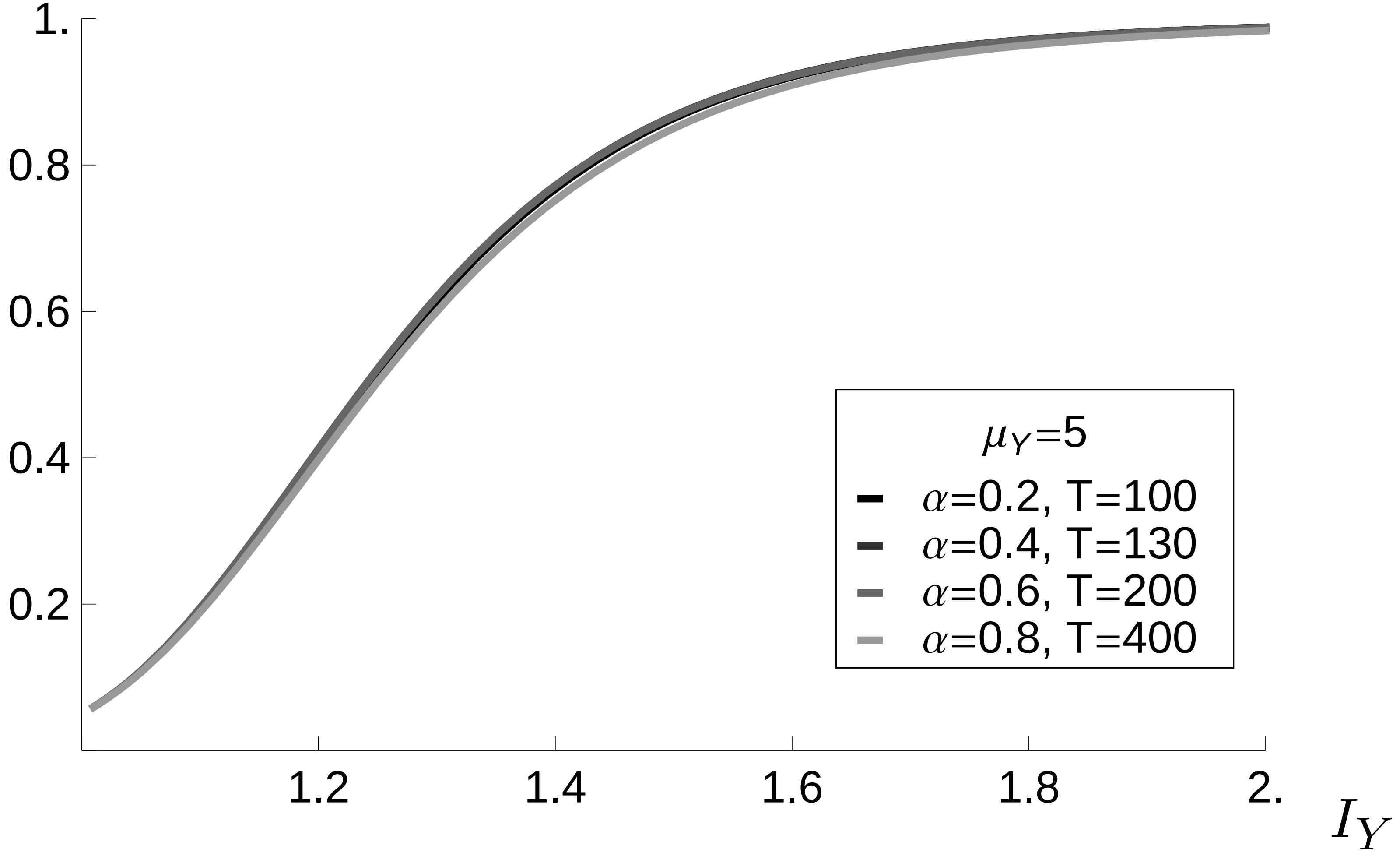
Power analysis for NB(n, π)-innovations:



Power analysis for NB(n, π)-innovations:



Power analysis for NB(n, π)-innovations:



Conclusions

. . . and Future Research

In a nutshell:

- CPINAR(1) model for overdispersed counts,
- asymptotic behaviour of CPINAR(1) model,
- asymptotic behaviour of index of dispersion, both for Poisson case (null) & overdispersed case (alternative),
- asymptotic power analysis: sensitivity of test severely influenced by serial dependence.

Ideas for future research:

- CPINAR(1): approximate marginal distribution for infinite compounding structure.
- CPINAR(1): parameter estimators and their (asymptotic) properties.
- Higher-order autoregressions $p > 1$, diagnosing $p > 1$.
- . . .

Thank You

for Your Interest!

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