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Bounded rationality in differential games

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Zusammenfassung / Abstract

The present paper proposes a myopic, boundedly rational heuristic for individual decision-making in differential game settings. I demonstrate that this type of behaviour converges to Nash equilibrium in infinitely repeated stage games without a state variable if the stage game is strategically symmetric. Two examples are used to illustrate the application of the heuristic in differential games.

JEL-Klassifikation / JEL-Classification: differential games, simulation, bounded rationality

Schlagworte / Keywords: C72

1 Introduction

Much of the economic analysis of conflict relies on game theory as an analytical tool.[1] When elements of control theory are integrated into a classical non-cooperative game, we obtain a *differential game* (Lewin 1994). These models are of high interest, but fraught with two problems: first, the resulting systems of differential equations frequently do not afford an analytical solution (Ruth and Hannon 2012), in particular if the equations of motion contain nonlinearities (Beckmann und Reimer 2014). Second, the calculations involved are so complex that it is hardly credible that real-world decision-makers would employ them at all. In the absence of repetition and learning, it is difficult to rationalise how agents could “feel their way” towards equilibrium.[2]

In the present paper, I address both of these problems by suggesting a boundedly rational heuristic for deciding in a differential game. This heuristic was used in Frey and Rohner (2007) as well as in Beckmann and Reimer (2014), but has not to my knowledge been analysed as general decision rule. As we will see in section 2, this heuristic also avoids the infinite utility problems that would ensue in an infinitely repeated game with no discounting.

I suggest that this decision rule is intuitively plausible and also simplifies both the calculations required of the agents and the resulting differential equations describing the behaviour of the system in such a way that analytical solutions can become available. This paper is organised as follows: The next section 2 sets out the idea for a simple decision heuristic and demonstrates how this would work in an infinite repetition of a simple stage game. Section 3 generalises the idea to differential games (with state variables) and considers two classical examples, the political business cycle and a rent-seeking game, to illustrate the usefulness of our approach. Section 4 concludes.

2 Basic approach

Consider a game with two agents $i \in \{1, 2\}$, each of whom has a compact set of pure strategies S_i. Denote strategy choices by σ_i and payoffs by $\pi_i(\sigma_i, \sigma_j)$ where j is the

[1] See Garfinkel und Skaperdas (2007) for an overview. Kress and Washburn (2009) present an instructive taxonomy of the relevant technical fields.

[2] Applications where optimal solutions are sought in order to program them into a computer, as in the pursuit-evasion models (Isaacs 1965), are a different matter altogether.

"other" player. The first order condition for an optimal choice given $\overline{\sigma}_2$ is the obvious

$$\frac{\partial \pi(\sigma_i, \sigma_j)}{\partial \sigma_i} = 0 \tag{2.1}$$

From the above, one can compute player i's reaction function by solving for σ_i and then find the Nash equilibrium (NE) as the solution of the system of the two reaction functions.

We now consider a simple dynamic version of the above game: strategy choices are made over a (possibly endless) interval of time $t \in [0; T]$. At each point in time, an identical version of the stage game is played; that is, the state of the system does not change as a consequence of agents' choices (an assumption we will relax in the next section). Let us use this simple scenario to posit a simple heuristic and explore its properties.

2.1 The decision heuristic

At any point of time, agent i takes j's current strategy σ_j as given (as in the Nash conjecture) and as time invariant — that is, agents are assumed to be *myopic*. We then assume the agent to calculate the optimal *static* response according to equation (2.1) and close a proportion α_i of the gap between this myopic optimum and its current strategy $\sigma_i(t)$. The equation of motion for i's control is given by

$$\dot{\sigma}_i = \alpha_i \left(\text{argmax}(\pi_i(\sigma_i(t), \sigma_j(t))) - \sigma_i(t)\right) \tag{2.2}$$

In addition to being myopic, this heuristic has two important properties:

1. Choices have a *history*, in particular, there is an exogenous starting strategy $\sigma(0)$. This can reflect such things as initial readiness or deployment in a theory of conflict.
2. Agents can differ in their *speed of adaptation* to the current myopic optimum. This property can be used to model differing speeds of decision-making processes or different flexibilities in implementing policy changes.

2.2 Convergence to Nash equilibrium

Denote i's the reaction function as $\sigma_i^* = \sigma_i^*(\sigma_j(t))$. Equation (2.2) can then be rewritten as

$$\dot{\sigma}_i = \alpha_i \left(\sigma_i^*(\sigma_j(t)) \ - \sigma_i(t)\right) \tag{2.3}$$

From total differentiation of i's first-order condition, we obtain

$$\frac{\partial \sigma_i^*}{\partial \sigma_j} = -\frac{\frac{\partial^2 \pi_i}{\partial \sigma_i \partial \sigma_j}}{\frac{\partial^2 \pi_i^2}{\partial \sigma_i^2}} = -\frac{d_{ij}}{d_{ii}} \tag{2.4}$$

We write d_{ij} as a shorthand for $\frac{\partial^2 \pi_i}{\partial \sigma_i \partial \sigma_j}$. Obviously, from the second order conditions for a maximum, we have $d_{11} < 0$ and $d_{22} < 0$, and the sign of the mixed partials depends on whether there is strategic complementarity or substitutability. Using (2.4), the Jacobian of our system is

$$J = \begin{pmatrix} -\alpha_1 & -\alpha_1 \frac{d_{12}}{d_{11}} \\ -\alpha_2 \frac{d_{21}}{d_{22}} & -\alpha_2 \end{pmatrix}$$

The eigenvalues of this Jacobian are given by

$$e_{1,2} = -\frac{1}{2}\left(\alpha_1 + \alpha_2 \pm \sqrt{\frac{4\alpha_1\alpha_2 d_{12} d_{21} + (\alpha_1 - \alpha_2)^2 d_{11} d_{22}}{d_{11} d_{22}}}\right)$$

Note that the reaction speeds α_i are non-negative. If the mixed partials have the same sign – i.e., if the game is weakly symmetric in the sense that the complementarity or otherwise of stategies is the same for both players –, then the eigenvalues will both be real-valued and no oscillations can occur. If the reaction speeds are identical, then $0 < d_{12}d_{21} < d_{11}d_{22}$ is a sufficient stability condition in the sense that both eigenvalues will be real-valued and negative.

3 Differential games

3.1 The state equation

The typical differential game model differs from the above game in that the players' *instantaneous utility* or payoff depends on the current value $x(t)$ of a *state variable* — or a vector of such variables —, while the change of the system state is a function of the system's history and the players' strategies or *controls*. Formally we have

$$\pi_i(t) = \pi_i(x(t)) \tag{3.1}$$

$$\dot{x} = f(x(t), \sigma_1(t), \sigma_2(t)) \tag{3.2}$$

We can translate this into the format of equation (2.1) by integration. Notice that

$$x(t) = \int_0^t f(x(t), \sigma_1(t), \sigma_2(t))\mathrm{d}t$$

to find

$$\pi_i(x_0, \sigma_1, \sigma_2) = \pi(\int_0^t f(x(t), \sigma_1(t), \sigma_2(t))\mathrm{d}t)$$

i's payoff at time t is therefore a function of the history of play and of the initial condition of the system x_0. We can still apply the heuristic represented by equation (2.2): players are assumed to treat the current state of the system $x(t)$ and their opponent's current strategies as given, calculate their optimal response in this static setting and then close a percentage α between the "best" response and their current strategy σ_i. The difference between the differential game and the repeated static game of the previous section is that the desired end state is now a moving target — the process no longer converges to the NE of the stage game, but to the NE of a static game described by *the system's state at termination* $x(T)$. (If time is infinite, the target can continue moving indefinitely.)

3.2 First example: the political business cycle

We now consider a simple application of our approach: the classical political business cycle model (Drazen 2000). There are two players, the government and the private sector. Let the short-run trade-off between unemployment $u(t)$ and inflation $p(t)$ be given by the simple Phillips curve

$$u(t) = \theta - (p(t) - p^E(t)) \tag{3.3}$$

where p represents the actual rate of inflation — the government's control — and p^E is the expected rate of inflation, controlled by the private sector. The government is assumed to minimise a quadratic loss function defined over the two bads inflation and unemployment:

$$w(t) = p(t)^2 + \kappa u(t)^2 \tag{3.4}$$

where κ is an exogenous weight increasing in the government's leftiness. As in the literature, we assume that the private sector's goal is to minimise the squared forecast error $(p(t) - p^E(t))^2$. The usual specification of adaptive expectations can be dropped because it is built into our behavioural assumptions as long as $\alpha < 1$.

As for the private sector, its optimal choice at each point of time (given a static reference) is obvious: it is to set $p^E(t) = p(t)$. However, only a proportion α of this gap can be closed. This yields the equation of motion

$$\dot{p}^E = \alpha(p(t) - p^E(t))$$

As for the government, we plug equation (3.3) into equation (3.4) to get

$$w(t) = (1+\kappa)p(t)^2 - 2\kappa p(t)(p^E + \theta) + \kappa p^{E2} + 2\kappa p^E\theta + \kappa\theta^2$$

The myopically optimal response is therefore

$$p(t) = \frac{\kappa}{1+\kappa}(p^E(t) + \theta)$$

which yields the following equation of motion (assuming for simplicity that the αs are the same):

$$\dot{p} = \alpha \left(\frac{\kappa}{1+\kappa} (p^E(t) + \theta) - p(t) \right)$$

These equations of motion can be studied in the usual way. First, we let $\dot{p}^E = \dot{p} = 0$ to find the steady state $p^{E*} = p^* = \kappa\theta$. We then compute the eingenvalues of the Jabobian

$$J = \begin{pmatrix} -\alpha & \alpha \\ \frac{\kappa\alpha}{1+\kappa} & -\alpha \end{pmatrix}$$

to find $e_{1,2} = -\frac{\alpha}{1+\kappa}\left(1 + \alpha \pm \sqrt{\kappa + \kappa^2}\right)$, both of which are real-valued and at least one of which is negative.The steady state will therefore be either stable or saddle point stable (if the signs differ).

3.3 Second example: a dynamic rent-seeking model

The second example extends the well-known basic model of rent-seeking (or, alternatively, wasteful military conflict) to a dynamic setting (Rowley, Tollison and Tullock 2013). Assume we have two agents vying for resources. At every point of time, agent i has $r_i(t)$ units of the resource pool under her control, which she can either invest into some productive activity or spend on fighting f. Denote the constant rate of return for peaceful production $\rho > 1$. The pool of resources that is up for grabs at time t is then $\rho \sum_j (r_j - f_J)$, and we assume that agent i receives a share p_i according to the standard *contest success function* (Hirshleifer 2001) $p_i = \frac{f_i}{\sum_j f_j}$. The equation of motion for i's resources is therefore

$$\dot{r}_i = \frac{f_i}{\sum_j f_j} \rho \sum_j (r_j - f_j) - r_i \tag{3.5}$$

In our bounded rationality model, agent i will choose f_i in such a way as to maximise this net gain of resources. The first-order condition for this problem simplifies to:

$$f_i \sum_j (r_j - f_j) + (\sum_j (r_j - f_j) - f_i) \sum_j f_j = 0 \tag{3.6}$$

For the remainder of this example, let us focus on the two-agent case. Using (3.6), we find that both agents aspire to realise $f_{1,2} = \frac{3}{8}(r_1 + r_2)$. We therefore have the following equations of motion for the controls:

$$\dot{f}_i = \alpha_i \left(\frac{3}{8}(r_1 + r_2) - f_i \right) \tag{3.7}$$

As the state and control variables differ in this model, the dynamics of the model come from a system of four ordinary differential equations – (3.5) and (3.7) – , half of which are non-linear. The nonlinearities lead to the usual kind of problems when attempting to solve the system explicitly.

In order to solve the equation system for a steady state, observe that we must have $f_1, f_2 \neq 0$ due to (3.5). From (3.7), we conclude that the r_is must also be nonzero. We proceed to proove the nonexistence of a steady state in three steps.

1. Consider a symmetric steady state with $f_1 = f_2 = f > 0$. We can then subtract the two versions of (3.5) to find that $r_1 = r_2 = r$. Substitute into (3.7) to find $f = \frac{3}{4}r$. Plugging this into (3.5), we find $r(\rho - \frac{3}{8}\rho - 1) = 0$, which implies that either $r = 0$ – which would lead to a contradiction – or $\rho = \frac{1}{1-3/8}$ and r free.
2. In any solution, we can add the two equations (3.5) to find, after some rearranging, that $(\rho - 1)(r_1 + r_2) = \rho(f_1 + f_2)$. Using (3.7), this in turn implies $(\rho - 1)(r_1 + r_2) = \rho\frac{3}{4}(r_1 + r_2)$ or $\rho = 4$.
3. Finally, our first (symmetric) version is a special case of the second, and so the second argument must apply to the first case also. However, ρ cannot simulataneously be equal to 4 and 1.6. Consequently, not steady state with nonzero values for r_i and f_i exists.

Consequently, numerical techniques must be used to explore the properties of this model. Figure 1 below shows a simulation model of equations (3.5) and (3.7) set up using *Stella*.[3]

It is easy to explore this model and to find that the typical results are not very sensitive to one's choice of parameter values. Figure 2 displays agent 2's fighting effort over time

[3] See `https://www.iseesystems.com`.

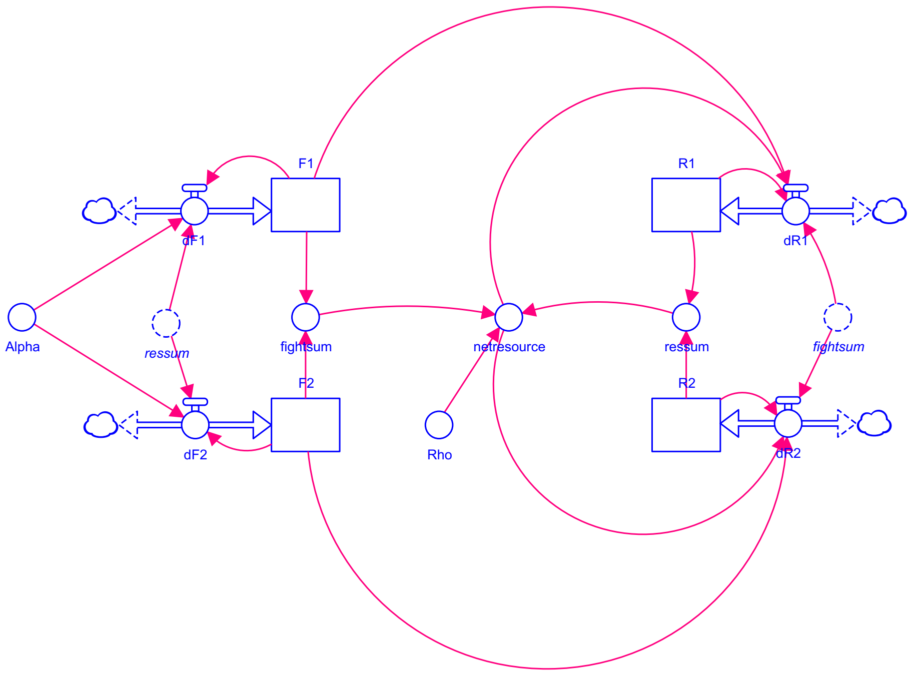

Figure 1: A *Stella* simulation of the dynamic rent-seeking model

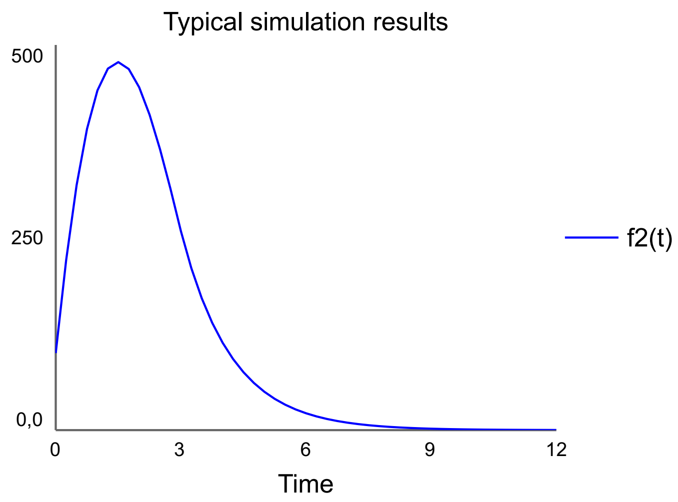

Figure 2: Results from a typical run of the simulation model

when we set $f_1(0) = f_2(0) = 100$, $r_1(0) = r_2(0) = 1000$, $\rho = 1.1$ and $\alpha_{1,2} = \frac{3}{4}$. We see that fighting intensifies in the beginning as the parties try to acquire the existing resources, but then decreases again as resources are depleted. Production fails to keep up with resource use for fighting, and the typical PD-type inefficiencies of rent-seeking models are clearly evident. In our model, *all* resources will be used up in the end. The system asymptotically approaches the origin ($f_1 = f_2 = r_1 = r_2 = 0$) without ever reaching it.

4 Conclusion

The present paper has shown how a boundedly rational heuristic can be used to simplify differential games. The basic assumptions resemble adaptive expectations as they were used in early dynamic models in the 1950s, and are admittedly rather simple. In contrast to these older models, they do contain an element of optimisation, and represent may be considered the "other extreme" when compared to the highly complicated calculations involved in solving differenctial games (if that is at all possible in a technical sense).

I presented to examples spanning the gamut of applications both regarding subject matter and our ability to solve the resulting system of ODEs. In the macroeconomic example, a steady states existed and its stability properties were amenable to analytical scrutiny. In the conflict / micro example, no steady states existed, and so simulation was used to describe the system's behaviour over time. However, in both examples we were able to state analytical equations of motion for the controls, which is more than can be said of many differential games with non-linearities.

References

Beckmann, Klaus B. and Lennart Reimer. "Dynamics of Military Conflict: An Economics Perspective." Review of Economics 65 (2014): 265–85.

Buchanan, JM, RD Tollison, and G Tullock. "Toward a Theory of the Rent-Seeking Society." Texas A & M Univ Pr (1980).

Drazen, A. "The Political Business Cycle After 25 Years." NBER macroeconomics annual (2000).

Garfinkel, Michelle R. and Skaperdas Stergios. "Economics of Conflict," In Handbook of Defense Economics, edited by Sandler Todd and Harley Keith, 649–709. Elsevier, 2007.

Hirshleifer, Jack. The Dark Side of the Force. Cambridge UP, 2001.

Isaacs, Rufus. Differential Games. John Wiley & Sons, Inc., 1965.

Lewin, Jospeh. Differential Games. London: Springer, 1994.

Matthias, Ruth and Hannon Bruce. Modeling Dynamic Economic Systems. Berlin: Springer, 2012.

Nordhaus, WD. "The Political Business Cycle." The review of economic studies (1975):

Rohner, Dominic and Bruno S Frey. "Blood and Ink! The Common-Interest-game Between Terrorists and the Media." Public Choice 133, no. 1-2 (2007): 129–45.

Rowley, C, RD Tollison, and G Tullock. The Political Economy of Rent-Seeking. 2013.

Ruth, Matthias and Bruce Hannon. Modellin Dynamic Economic Systems. Springer: Berlin. 2012.

Tollison, RD. "Rent Seeking: A Survey." Kyklos (1982).

Tullock, Gordon. “The New Palgrave Rent Seeking,” edited by Eatwell John, Palgrave Macmillan, 1987.

Washburn, Alan, R. and Kress Moshe. Combat Modeling. Berlin: Springer, 2009.

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