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A critique of the Richardson equations

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Zusammenfassung / Abstract

We review, and extend, one of the classic dynamic models of conflict in economics by Richardson (1919) and Boulding (1962). Restrictions on parameters are relaxed to account for alliances and for peace-keeping, yielding new dynamic patterns of conflict. In addition, we explore an incrementalist version of the model as well as a stochastic one and show how this affects its theoretical stability properties. Using Monte Carlo techniques as well as time series analyses based on GDELT data (for the Ethiopian-Eritreian war, 1998-2000), we also assess the empirical usefulness of the model. It turns out that the simulations fail to converge in a large number of cases, and that one important prediction of the model is not borne out by the data. We therefore conclude that the Boulding-Richardson equations are of limited use for modelling (de-)escalation in dynamic conflict.

JEL-Klassifikation / JEL-Classification: B25, D74

Schlagworte / Keywords: Conflict dynamicy, patterns of conflict, GDELT, time series, differential equations

1. Introduction

The formal, quantitative analysis of dynamic conflict in economics began a century ago, when Frederick Lanchester (1916) developed his linear and square laws of attrition. Such models consist of systems of ordinary or partial differential equations, whose dynamic behaviour and stability properties are analysed by solving the systems analytically or by phase diagramme techniques. While optimising agents may be embedded in larger attrition models (Washburn and Kress, 2009, ch. 5), this class of model typically does not contain explicit optimisation. Relevant models include the Lotka-Volterra-Goodwin equations of predator-prey conflict (Goodwin, 1967), the Intriligator and Brito (1986) model and the (Richardson, 1919) equations, which were extended in monographs published posthumously (Richardson, 1960a,b) as well as in Boulding (1962).[1]

The absence of explicit optimisation led to a shift of interest away from the first generation of models and towards dynamic games and, in particular, differential games (Isaacs, 1954). As far as static patterns of conflict are concerned, game theory provided a convincing taxonomy (Rapoport et al., 1976),[2] and dynamic game theory yielded deep insight into such features of conflict dynamics as the initiative, signals, and reputation. However, a general taxonomy of conflict dynamics proved elusive, and differential game theory hit a conceptual wall when dealing with the non-linearities that are pervasive in conflict theory (Beckmann and Reimer, 2014). These problems as well as improvements in the raw computing power available to scholars led to increased reliance on simulation methods (Fontana, 2006). And for simulation purposes, both the aforementioned theoretical limits and the insights of behavioural economics recommend some version of boundedly rational optimising. It is in this context that first-generation models may return as more than just a subject for the historian of economic thought.

In the present paper, we propose to re-visit the Richardson equations, discussing possible extensions as well as conducting simulation runs and estimating the model empirically. We intend to provide a full critique of this model, demonstrating its potential to describe intersting dynamic patterns of conflict as well as a severe empirical weakness.

[1] Lewis Fry Richardson is little known in economics, and without the work of his fellow Quaker Boulding his work may not have resounded in our field at all. He is, however, well remembered for his contributions to other disciplines. On this, see Hunt (1995). In mathematics, Richardson's equations are a popular simple model of conflict, which amongst other things is used in the classroom to explain phase diagramming, see `https://www.youtube.com/watch?v=e3FfmXtkppM`. The recent conflict economics text by Anderton and Carter (2009) also has a section on the Richardson model.

[2] Cf. the recent book by Robinson and Goforth (2005).

The first part of this paper (section 2) is theoretical in nature. We begin with the standard two-party case (subsection 2.1). Our focus in this section is on different ways to model the interaction term between force (or escalation) levels and to show how this choice affects the (theoretical) stability properties of the model. While the majority of work in conflict economics is couched in terms of two-player models,[3] note that the three-plus-case differs from the two-party one in a very important way, namely that one of the parties can act as an *attenuator*, trying to dampen the conflict between the remaining agents (Goldstein and Pevehouse, 1999). Richardson (1960a) already tackled 3-party and n-party cases, but constrained all parameters to be positive, which curtails the additional possibilities in a three-way interaction. Sub-section 2.4 addresses this omission, dealing with *alliances* and *conflict attenuation* in particular. We also consider Boulding (1962)'s *psychological* interpretation of the Richardson equations, arguing that the existing model (irrespective of the number of dimensions) does not fit this story well. An extended version is developed for this purpose.

In the second half of this paper (section 3), we move from theory to simulation studies and empirical analysis. We first return to the two-party setting and illustrate the applicability of the Richardson model using GDELT data on the Ethiopian-Eritreian war of 1998-2000 (sub-section 3.1). We then explore our own extension of the three-party model using computer simulation and Monte Carlo methods, studying the frequency with which convergence occurs (sub-section 3.2). Both empirical and simulation analyses do not bode well for the Richardson family of models, a finding that we comment on in the conclusion (section 4).

2. Richardson's theory and extensions

Richardson (1919) conceived of his equations as a model of an arms race (see also Anderton and Carter, 2009, pp. 199-202). In Boulding (1962)'s version, the equations describe the joint dynamics of the *aggressiveness* (or escalation level) of two parties to a conflict. We will use both stories interchangeably at first, but return to the difference between the two in sub-section 2.7.

[3] Two-party interactions are prevalent among the models presented in Hirshleifer (2001) as well as in the volumes edited by Sandler and Hartley, eds (1995, 2007) or Garfinkel and Skaperdas, eds (2012).

2.1. Richardson equations in two dimensions

Denote by a (b) a measure of party A's (B's) armament, or aggressiveness towards the other, and assume that without interaction, this reverts over time to a base level $\hat{a}$ ($\hat{b}$). This base level is, however, not the long-term equilibrium because of the interaction effect: each party's aggressiveness increases exponentially as a function of the competitor's escalation measure. Together with the assumption $a, b > 0$, this gives the Richardson equations

$$\dot{a} = k_a(\hat{a} - a) + r_a b \tag{1}$$

$$\dot{b} = k_b(\hat{b} - b) + r_b a \tag{2}$$

where the strictly positive parameters k_i and r_i represent the parties' speed of adjustment to the base level and sensitivity to aggression, respectively.

We can explicitly solve this system of linear ODEs for the time paths $a(t), b(t)$ of aggressiveness. For example, in the symmetric case where $r_a = r_b = r$ and $k_a = k_b = k$ (assuming $r \neq k$), we find

$$a(t) = \frac{k(\hat{a}k + \hat{b}r)}{k^2 - r^2} + \mathrm{e}^{-kt}(c_1 \cosh(tr) + c_2 \sinh(tr)) \tag{3}$$

where c_1 and c_2 are constants. If we additionally assume that $a(0) = b(0) = 0$, we have

$$a(t) = \frac{e^{2rt} - e^{(k+r)t}}{\frac{r-k}{k} e^{(k+r)t}} \hat{a} \tag{4}$$

and likewise for b.

However, the general properties of this model are better studied using phase diagramme techniques. Letting $\dot{a} = 0$ and $\dot{b} = 0$, we obtain the "nullclines" where the vector field is vertical and horizontal, respectively (written as functions of a for easier plotting)

$$b = \frac{k_a}{r_a}(a - \hat{a}) \tag{5}$$

$$b = \hat{b} + \frac{r_b}{k_b} a \tag{6}$$

Note that both graphs are upward sloping lines in (a, b)-space and that the equation for $\dot{a} = 0$ has a negative intercept on the b axis, while the other cuts the ordinate at $\hat{b} > 0$. This already implies that there are just two possible configurations (see figure 1). If $\frac{k_a}{r_a} > \frac{r_b}{k_b}$, the two lines intersect in the positive orthant (left-hand panel in figure 1) and there exists a stable stationary equilibrium at

$$(a^*, b^*) = \left(\frac{k_b(\hat{a}k_a + \hat{b}r_b)}{k_a k_b - r_a r_b}, \frac{k_a(\hat{b}k_b + \hat{a}r_b)}{k_a k_b - r_a r_b} \right) \tag{7}$$

Otherwise, there is no such intersection and aggressiveness explodes in the positive orthant (right-hand panel in figure 1). Observe that a *symmetry* assumption, i.e. $r_a = r_b$ and $k_a = k_b$, would generate a borderline case where the graphs of (5) and (6) are parallel. The consequences are much the same as in case 2 above, with an explosion of aggression in the first orthant.

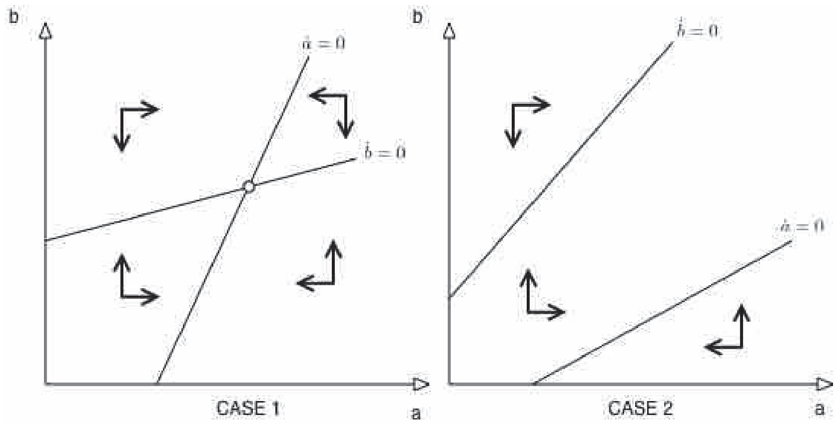


Figure 1: The two possible scenarios in the B-R model

Formally, note that the Jacobian for the system (1) and (2) is

$$J = \begin{pmatrix} -k_a & r_a \\ r_b & -k_b \end{pmatrix}$$

with the two eigenvalues $\lambda_{1,2} = -\frac{1}{2}(k_a + k_b \pm \sqrt{(k_a - k_b)^2 + 4r_a r_b})$. As the term under the square root must be positive given our assumptions, both eigenvalues are real. The obvious condition for both eigenvalues to be *negative* is

$$k_a + k_b > \sqrt{(k_a - k_b)^2 + 4r_a r_b}$$

Square both sides of this inequality and rearrange to find $k_a k_b > r_a r_b$, which is equivalent to the graphical restriction on slopes given earlier as a condition for stability. If this inequality does not hold, we will have two real eigenvalues with differing signs, i.e. saddlepoint stability (however, the equilibrium will be in the negative orthant).

The endless escalation of conflict in this case (2) may appear implausible because infinite aggression levels are an unwieldy concept. However, in interpreting the model, one can assume that there exists a threshold level of escalation beyond which the conflict in question changes its nature (i.e., an open outbreak of military hostilities). One can also add an additional constraint to the model – for instance, a and b could represent the *share* of two competing news media (total broadcast time or pages in a magazine) devoted to a particular conflict, or a particular scandal. The latter modification would give rise to a stable corner solution.

We now propose two variants of the two-agent Richardson model, which we explore in turn:

1. a version which incorporates the idea that it may be escalation rather than the stock of aggressiveness which determines the interaction effect,
2. a model which replaces the deterministic interaction effect with a probabilistic version, taking account of Clausewitzian friction and other sources of uncertainty.

2.2. An incrementalist Richardson model

In our first variation, we recognise that it can be the *change* in enemy aggression levels, i.e. the *escalation* of conflict, which drives conflict dynamics. We retain the assumption that aggression levels will return to base values $\hat{a}, \hat{b}$ over time, but replace the stock levels of aggression with their time derivatives $\dot{a}, \dot{b}$. This leads to the following model:

$$\dot{a} = k_a(\hat{a} - a) + r_a \dot{b} \tag{8}$$

$$\dot{b} = k_b(\hat{b} - b) + r_b\dot{a} \tag{9}$$

As was the case for the baseline model, we can solve this system of differential equations explicitly, obtaining complete time paths for the two variables of interest, given the parameters and starting values $a(0), b(0)$. Using the symmetric example from section above, we find

$$a(t) = \left(1 - e^{\frac{k(1+r)t)}{r^2-1}}\right) a(0) \tag{10}$$

with an analogous solution for b. Again, however, we find it more instructive to take a conventional approach using phase diagrammes to illustrate system behaviour over time for more general parameter values.

Substituting $\dot{b}$ into the first equation of the model and rearranging, we can express the change in a and in b as a function of the state variables

$$\dot{a} = \frac{k_a(\hat{a} - a) + r_a k_b(\hat{b} - b)}{1 - r_a r_b} \tag{11}$$

$$\dot{b} = \frac{k_b(\hat{b} - b) + r_b k_a(\hat{a} - a)}{1 - r_a r_b} \tag{12}$$

Proceeding as before, we obtain the following equations for the nullclines:

$$b = \frac{\hat{a}k_a + \hat{b}k_b r_a - k_a a}{k_b r_a} \tag{13}$$

$$b = \frac{\hat{b}k_b + \hat{a}k_a r_b - k_a r_b a}{k_b r_a} \tag{14}$$

Solving this simple system yields the stationary point at $a^* = \hat{a} \wedge b^* = \hat{b}$. This implies that contrary to the standard B-R model, the stationary point always lies in the positive orthant.

For a graphical analysis, observe that the slope of the graph for $\dot{a} = 0$ is steeper than the other iff $r_b < 1$. Also note that the denominator in both equations of motion (11) and

(12) becomes *negative* for $r_a r_b > 1$. All in all, this leaves us with four possible dynamic configurations shown in figure 2 below. Case 1 exhibits a stable stationary state, whereas case 2 is characterised by instability. However, case 2 differs from the unstable case in the original model in that a corner solution at the origin is also a possibility. Cases 3 and 4 – where $r_b > 1$ – have saddlepoint stable equilibria.

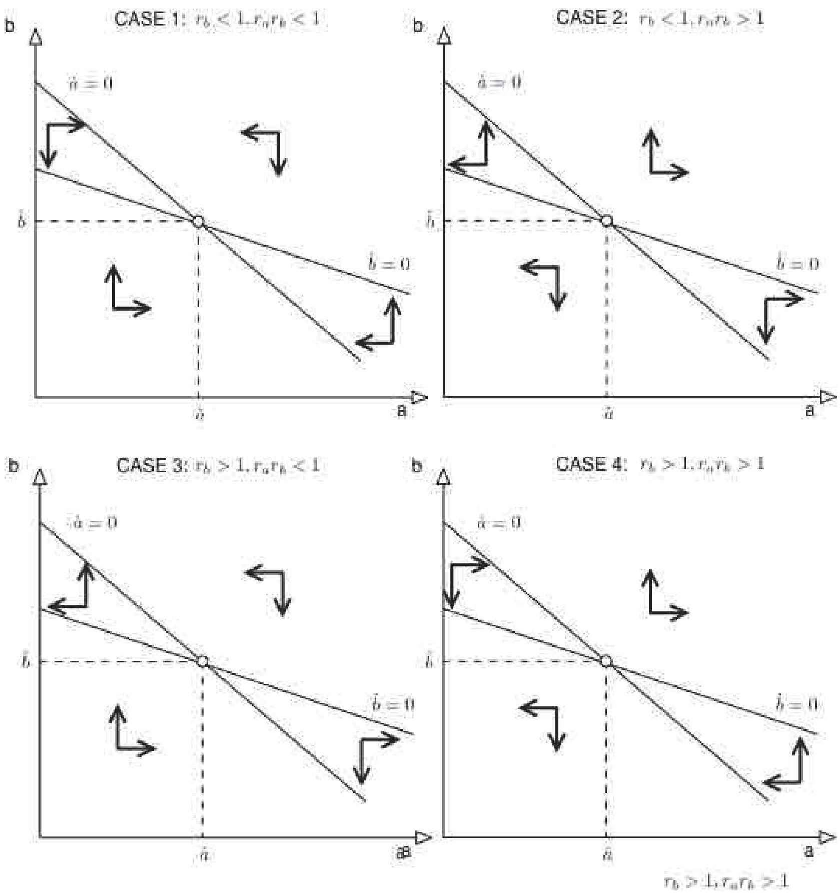


Figure 2: The four scenarios in the incrementalist Richardson model

Start from equations (11) and (12) to find the Jacobian

$$J = \begin{pmatrix} -\frac{k_a}{1-r_a r_b} & -\frac{r_a k_b}{1-r_a r_b} \\ -\frac{r_b k_a}{1-r_a r_b} & -\frac{k_b}{1-r_a r_b} \end{pmatrix}$$

and the two eigenvalues $\lambda_{1,2} = \frac{k_a+k_b\pm\sqrt{(k_a-k_b)^2+4k_ak_br_ar_b}}{2r_ar_b-2}$. While we can rule out complex eigenvalues again, the fact that the sign of the denominator reverses at $r_ar_b = 1$ now gives rise to a total of four possible configurations, as shown in figure 2 above.

In the original Richardson model, it was the *relative* size of adaptation k and reaction coefficients r that determined the dynamic pattern of conflict. Now, it is the *absolute* value of the reaction coefficients alone that proves crucial. It is sufficient for convergence to a stable equilibrium at the "normal" aggro level $\hat{a}, \hat{b}$ that both parties do not respond "in kind" to an enemy escalation, but with an $r < 1$. This feature of the model appears more plausible than the results we obtained for the original formulation. In addition, the incrementalist model allows for a "pacifist" party (with low r) to compensate for the existence of an aggressive opponent in a very plausible manner.

2.3. Probabilistic interaction

Finally, let us briefly consider how to incorporate *randomness* – and Clausewitzian "friction" – into the simple framework. As a large conflict unfolds, there will be several small interactions during which either side can either escalate, de-escalate, or ignore the other side's aggression. Let a's probability p of escalation depend on b's aggro level according to a probability function $p(b)$ with $p' > 0$ and vice versa. For a large number of such interactions per unit of time, the equations of motion can then be amended by just plugging in the probability functions for r_ab and r_ba, respectively. We then obtain the following system of equations

$$\dot{a} = k_a(\hat{a} - a) + s_ap(b) - s_a(1 - p(b)) \tag{15}$$

$$\dot{b} = k_b(\hat{b} - b) + s_bp(a) - s_b(1 - p(a)) \tag{16}$$

where the s_i represent party i's "step size" of (de-)escalation, assumed to be a constant for simplicity.

We require a specific probability function for plotting or explicit solutions, although basic phase diagrammes such as the ones in figures 1 and 2 could by derived with just some assumptions regarding the curvature of p. Borrowing from the literature on conflict success functions,[4] we employ a *logistic function*

[4] The classic treatment is the book by Hirshleifer (2001).

$$p(a) = \frac{1}{1 + e^{\kappa(\hat{a} - a)}} \tag{17}$$

where $\hat{a}$ denotes the reference level of aggression by A (i.e., the level where escalation and de-escalation are just as likely), and the parameter κ determines the steepness of the probability function.

One important difference from the variants discussed previously is that the isoclines for $\dot{a}, \dot{b} = 0$ are now non-linear. Also, the fact that the limits of the logistic function are zero for $a, b \to -\infty$ and one for $a, b \to +\infty$ together with the structure of the system imply that there exists a stable intersection in the positive orthant. Figure 3 below illustrates this for the symmetric case.[5]

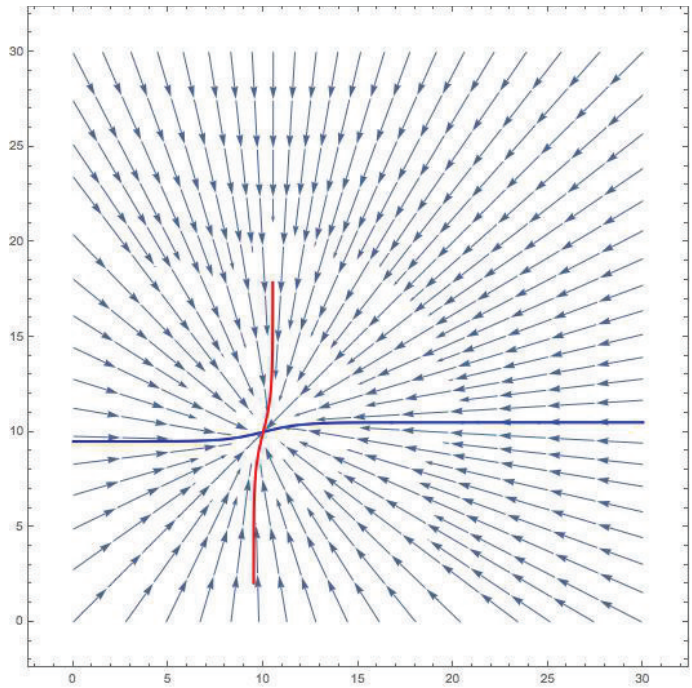


Figure 3: A stable stationary point in the probabilistic model

[5] We assume a symmetric solution with the following parameter values: $k_a = k_b = \frac{1}{5}, r_a = r_b = \frac{1}{2}, \hat{a} = \hat{b} = 10$. The plot was produced using *Mathematica*.

2.4. More than two parties

Consider the Richardson equations when there are three parties to the conflict (Richardson, 1960a), whose respective state variables are a, b and c:

$$\dot{a}_t = k_a(\hat{a} - a_t) + r_{ab}b_t + r_{ac}c_t \tag{18a}$$

$$\dot{b}_t = k_b(\hat{b} - b_t) + r_{ba}a_t + r_{bc}c_t \tag{18b}$$

$$\dot{c}_t = k_c(\hat{c} - c_t) + r_{ca}a_t + r_{cb}b_t \tag{18c}$$

The first term on the right-hand sides of the above equations reflects a tendency for the state to return to an exogenous "normal" level, with the parameter $k_i > 0$ incorporating the speed at which this happens to actor i. The parameter r_{ij} denotes party i's *reaction coefficient* to party j's state. As is usual, we will drop time indices throughout the following discussion whenever we can do so without ambiguity.

In Richardson (1919)'s original account, the state variables represent the levels of armament of state actors involved in an arms race. The reaction coefficients r_{ij} are all positive as higher levels of armament on the part of other governments induce additional military procurement, *ceteris paribus*.

The computation of a stationary point for the above system is straightforward and leads to simple but clumsy expression, which we will not display here. The Jacobian for the system is

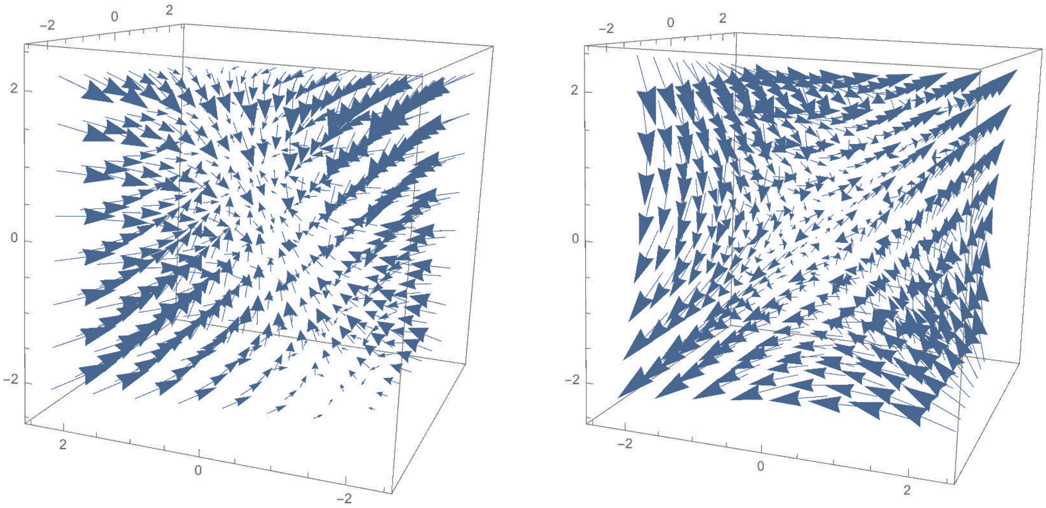
$$J = \begin{pmatrix} -k_a & r_{ab} & r_{ac} \\ r_{ba} & -k_b & r_{bc} \\ r_{ca} & r_{cb} & -k_c \end{pmatrix}$$

with the characteristic equation

$$\lambda^3 + (k_a + k_b + k_c)\lambda^2 + (k_a k_b + k_a k_c + k_b k_c - r_{ab}r_{ba} - r_{ac}r_{ca} - r_{bc}r_{cb})\lambda + \gamma = 0$$

where

$$\gamma = k_a k_b k_c - k_c r_{ab} r_{ba} - k_b r_{ac} rca - k_a r_{bc} r_{cb} - r_{ac} r_{ba} r_{cb}$$



(a) A stable node (b) Instability in the positive orthant

Figure 4: Richardson's original dynamic configurations in the 3-agent case

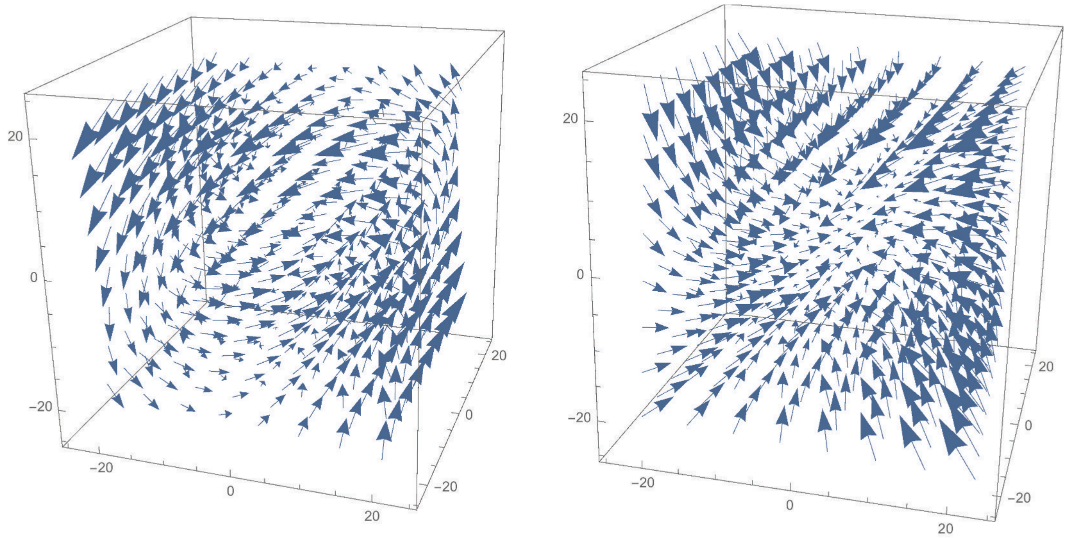
It is a tedious exercise to solve this for the eigenvalues of J. The equation has three solutions, one of which is real and two of which are complex. The signs of the real parts can be both positive and negative depending on the parameters of the system, in particular on the relative sizes of the k's and r's. The two diagrammes in figure 4 on page 12 depict the dynamic behaviour of the system under the assumption that all r_{ij}'s and k_i's are the same. As in the two-dimensional case, we find a stable node when $k > r$ and an unstable node otherwise:[6]

Our first extension of this original Richardson model is based on the idea of admitting *negative* parameters r_{ij}. This leads to two interesting configurations detailed below. But note that these extensions are only possible if there are three or more parties to the conflict – the concentration on the two-agent case hitherto obscured this.

2.5. Attenuating agents

Let agent c be an "attenuator", which we define as a player whose activities *reduce* the armament / escalation level of both other parties to the conflict. An example (hopefully)

[6]In the first example, we assume that $r = \frac{1}{20} \wedge k = \frac{1}{10}$, in the second example, we have $r = \frac{1}{10} \wedge k = \frac{1}{20}$.



(a) A center

(b) Stable node with asymmetric patterns

Figure 5: Additional dynamic patterns in the presence of attenuating agents

would be NATO engaged in a peace keeping mission (Goldstein and Pevehouse, 1999). The state variable c can now be interpreted as an *involvement level*, which depends positively on the level of armament / aggression by the other parties, i.e. $r_{ca}, r_{cb} > 0$. However, $r_{ic} < 0$ for all $i \in (a, b)$.

Admitting negative values for some parameters opens up several new possibilities. For example, the following vector plot shows the case where $k_i = \frac{1}{20} \forall i$ and $|r_{ij}| = \frac{1}{10}$, but c's influence on the other two players is attenuating. This leads to the emergence of a *center* with cyclical trajectories and periodic motion of the armament levels, which has not so far been described in the literature on Richardson-type models. See the left panel in figure 5 on page 13.

Stable nodes also can display more complex dynamic behaviour than in the two-party case. For example, in the special case where $k_a = k_b = k_c$ and $r_{ca} = r_{cb} = 0.2$ while $r_{ab} = 0.3$, $r_{ba} = 0.1$ as well as $r_{ac} = r_{bc} = -0.2$, the numerical eigenvalues (up to three digits' precision) are $\lambda_1 = -0.688$, $\lambda_2 = -0.406 + 0.276i$ and $\lambda_3 = -0.406 - 0.276i$. The resulting asymmetric trajectories are illustrated in the vector plot in sub-figure 5b on page 13.

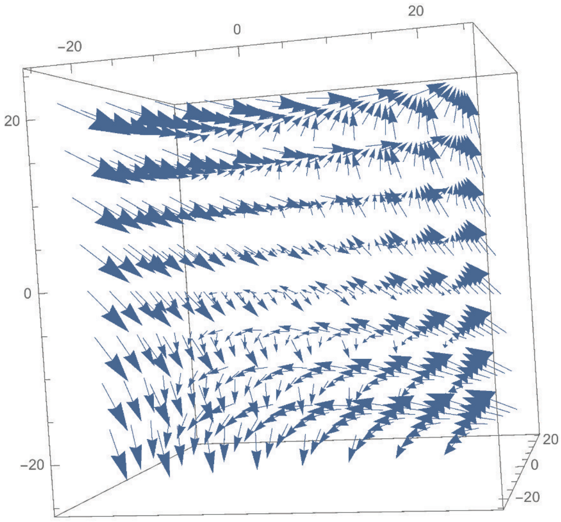


Figure 6: An unstable vortex in the presence of alliances

2.6. Alliances

In an alliance, higher armament by one of the parties will lead – all other things being equal – to a reduction of effort by the other due to the familiar incentive to free-ride on the contribution of others (Peinhardt and Sandler, n.d.; Sandler, 2004). That is, if a and b are allies in an otherwise standard conflict with c, we would expect $r_{ab}, r_{ba} < 0$, while all other coefficients remain positive.

Numerical experimentation using our standard example (where the $k_i = 0.05$ and $|r_{ij}| = 0.1$) reveals that the system may now behave as an unstable vortex, with three imaginary eigenvalues, two of which have negative real parts. Figure 6 on page 14 below shows this particular example.

2.7. A psychological interpretation of the Richardson equations

Boulding (1962) offered a *psychological* story to tell with the Richardson equations. In his version, the state variables measure the *hostility* of one party towards the other – in video game parlance, their "aggro levels". There is a tendency for the hostility level to regress to a base value $\hat{a}, \hat{b}$, and one party will escalate when their adversary's hostility towards them increases. This appears to be a natural interpretation of the equations in the two-agent case. But consider the system (18) in matrix form (we assume $\hat{a} = \hat{b} = \hat{c}$ for simplicity):

$$\begin{pmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} -k_a & r_{ab} & r_{ac} \\ r_{ba} & -k_b & r_{bc} \\ r_{ca} & r_{cb} & -k_c \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{19}$$

This equation shows that the Richardson equations *force an agent's hostility towards the other two to be the same.* There is also a nonnegativity restriction on the state variables which makes perfect sense when modelling arms races – where negative force levels are an impossibility –, but seems unsatisfactory in international relations, where there may exist sympathy, and where enmity can transmogrify into friendship (at least if you subscribe to idealism).

We now propose an extension of the model that removes these drawbacks. In doing so, we combine three ideas developed in the preceding sections:

1. Agents' attitudes towards other agents may differ, and they may be negative (friendship).
2. The nonnegativity restriction is dropped on all parameters except the one reflecting the rate of reversion to the "natural" aggressiveness level. This allows for attenuating agents as well as for alliances (see sub-sections 2.5 and 2.6).
3. The interaction effect is taken to depend on the *escalation* of aggression experienced by an agent rather than its level (see sub-section 2.2).

The state of the system is now represented by a *two-dimensional matrix* $\mathcal{S}$ of size $n \times n$ rather than a vector, where n denotes both the number and the set of agents. Each element s_{ij} of $\mathcal{S}$ measures agent i's hostility towards j. The case where $i = j$ is considered as a general level of hostility of agent i which is not directed towards another agent in particular (and treated differently). We impose no *a priori* restrictions on s_{ij} as we want to permit empathy as well as hostility (where a negative s refers to the former in line with Goldstein (1992) scoring). It follows that the matrix of changes will also be of dimension $n \times n$. The second building block of the model is a $n \times n \times n$ matrix $\mathcal{A}$ of parameters capturing the responsiveness to others' aggression levels. Each element a_{ijk} of $\mathcal{A}$ represents the change in agent i's attitude towards agent j as a result of agent k's hostility level. Accepting that we are limiting ourselves to exponential processes (linear changes in time), this framework is the most general one.

The n^2 equations of motion in this system are the following:

$$\dot{s}_{ij} = a_{iji}\left(\hat{s}_i - s_{ii}\right) + \sum_{k \neq i}^{n} a_{ijk}\dot{s}_{ki} \quad \forall \quad i, j \in n \tag{20}$$

Equation (20) – with $a_{iji} > 0$ – reflects the standard property of the Richardson model that there exists a "natural" level of aggression $\hat{s}_i$ for each country i to which it would revert in the absence of any interaction between countries. It also captures these interactions, stipulating that country i's hostility (or empathy) towards country j changes as a result of the change in the hostility *other* countries exhibit towards i, where there is a k-specific linear effect. The appendix (section 4) contains an implementation of the model in the Wolfram Language[7] that will be used for our Monte Carlo study of the model's properties in section 3.2.

3. Empirical suitability

We now proceed to check the empirical applicability of the Richardson model, beginning with an empirical illustration of the two-agent case. In the final sub-section, we turn to Monte Carlo simulations of a three-agent version of the new model we proposed as a consequence of our theoretical discussion.

3.1. A case study for the two-agent case

In order to provide an empirical illustration for the suitability (or otherwise) of the Richardson model for two parties, two obvious prerequisites need to be met: first, one has to find a well-documented conflict in history with just two parties to it, and second, the state variables of the model need to be identified in the appertaining dataset.[8]

Regarding the first issue, we focus on the war between Ethiopia and Eritreia (hostilities lasted from May 1998 to May 2000,[9] but our dataset includes the three years preceding the outbreak of hostilities and following the ceasefire), arguing that this is indeed a conflict in which external players and mediators did not play a decisive role.

[7] See `wolfram`.

[8] We recognise in passing that Richardson (1960b) also played a pioneering role in the systematic collection of data about conflicts.

[9] See `https://en.wikipedia.org/wiki/EritreanEthiopian_War`.

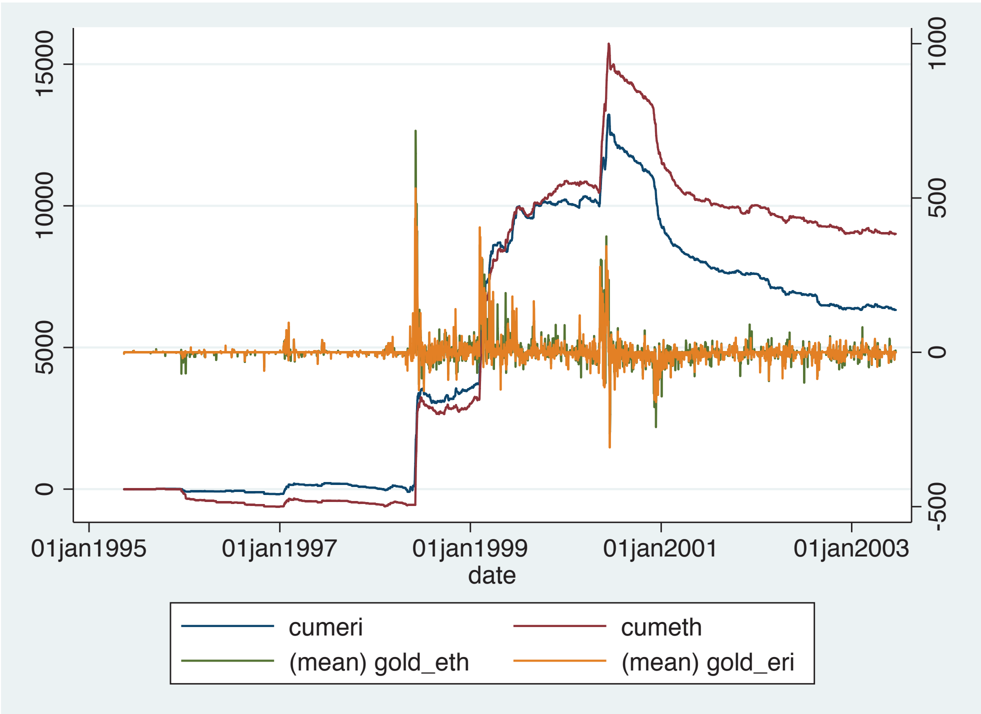


Figure 7: Goldstein (1992) scores and "Goldstein levels" for the Ethiopian-Eritreian conflict, 1995-2002

As to the second, we follow a large strand of the literature using the GDELT[10] database of coded discrete event data (Goldstein and Pevehouse, 1999). Our measure of *(de-)escalation* or the *change of aggressiveness* is the Goldstein score (Goldstein, 1992), which assigns each conflictary (and cooperative) action an integer in the interval $[-10;10]$ indicating the flow impact on relations between the involved parties. The state variables, therefore, are just the sum of the (undiscounted) Goldstein scores accumulated over the course of the conflict. One unfortunate consequence of this is that we need to fix the starting values for the state variables $a(0), b(0)$ at some arbitrary level – zero in the following illustration. This being said, figure 7 above plots the four time series – our measures for $\dot{a}, \dot{b}, a, b$ – over time.[11]

On inspection of figure 7, we observe that the accumulated Goldstein scores remained constant in the years preceding the war and abated in the aftermath of the ceasefire, we also observe a (not unexpected) steep increase during hostilities. The change of aggressiveness during the war is dominated by three extreme peaks corresponding to major campaigns[12] and also influenced by the onset of the rain season, which impeded the movement of motorised troops. It is clear that such peaks of escalation are incompatible with the Richardson model. This bolsters Richardson's original arms race story relative to Boulding's psychological version, which might be applied to wartime aggression as well. It is also compatible with the hypothesis that open hostilities arising whenever the stock of aggression exceeds an exogenous threshold level in an unstable Richardson model.

Evidence of an additional difference between the shooting war and the period of time preceding it can be found by looking at the correlograms of the Goldstein score time series separately for the war and the three years leading up to it (figure 8). Not only does there seem to be more autocorrelation during the war, but significant lags appear clustered over the first week. This is suggestive of the effect of military planning leading to continuous activity. During the crisis before the war, on the other hand, significant lags were not concentrated in the same manner, and the correlation coefficients do not shrink over time as they do during the war.

Having observed differences between the wartime and pre-/post-war times series and

[10]The GDELT Project – Global Database of Events, Language and Tone – `http://gdeltproject.org/`.

[11]The two flow variables are called "golderi" – the sum of the Goldstein indices assigned to Eritreia's actions towards Ethiopia on a given day – and "goldeth", while we refer to the stock variables as "cumeri" and "cumeth".

[12]The Eritrean attack on Badme in May 1998 including the subsequent air war, Ethiopia's offensive of February 1999, and the final Ethiopian attack in May 2000 that severed Eritreian lines of communication and paved the way for the ceasefire.

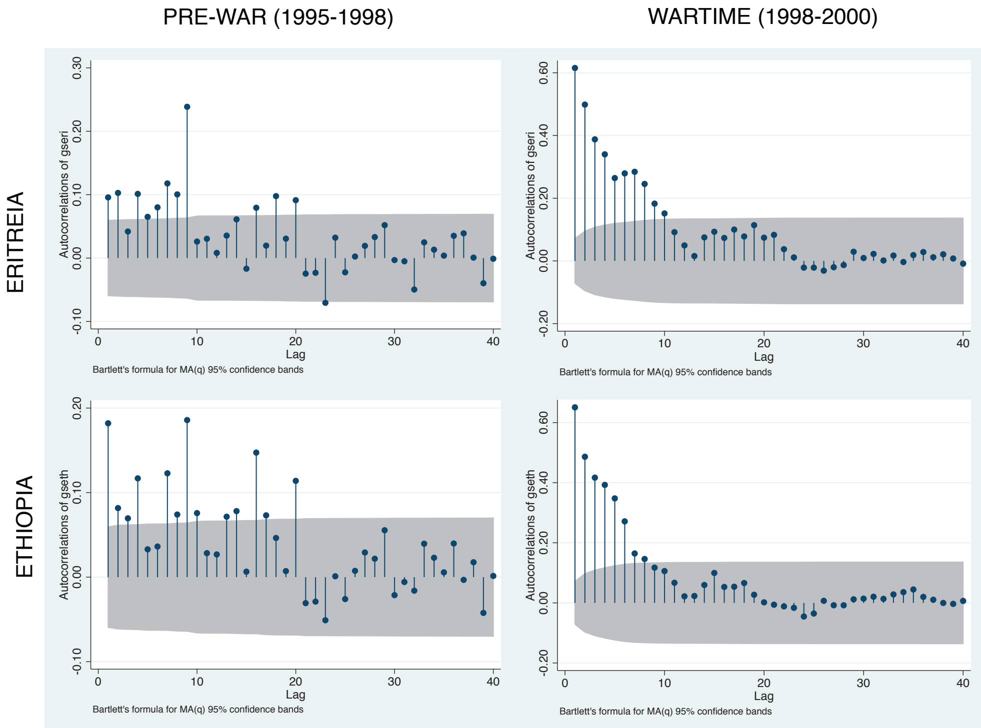


Figure 8: Correlograms for our time series of Goldstein scores

Table 1: Summary of time series analyses

Type of analysis	Time series	1995-1998	1998-2000	2000-2003
Stationarity	goldeth	yes***	yes***	yes***
	golderi	yes***	yes***	yes***
	cumeth	no	no	no
	cumeri	no	no	no
Cointegration	golderi, goldeth	yes***	yes***	yes***
	cumeri, cumeth	no	no	yes***
Granger causality	golderi $\Rightarrow$ goldeth	yes***	yes***	yes***
	goldeth $\Rightarrow$ golderi	yes***	yes***	yes***
	cumeth $\Rightarrow$ golderi	yes***	yes***	yes***
	cumeri $\Rightarrow$ goldeth	yes***	yes***	yes***
	cumeth $\Rightarrow$ goldeth	no	no	no
	cumeri $\Rightarrow$ golderi	no	no	no

recognised that the B-R model appears comparatively less attractive as a framework for modelling the former, we now proceed to a formal time series analysis of our data set. Table 1 on page 20 summarises the results. An augmented Dickey-Fuller test using 21 lags as suggested by the Schwert criterion allows us to reject the null of a unit root for our time series of Goldstein scores (analysing the pre-war, wartime and post-war periods separately). Unsurprisingly, this is not the case for the state variable, i.e. the accumulated scores (see table 1). We use a Johansen test for cointegration – again with 21 lags – and find that the time series for the Goldstein scores are clearly cointegrated at all conventional levels of significance, while no significant evidence of cointegration can be found for the accumulated scores with the interesting exception of the post-war period.

The most interesting (non-)results can be found in the last two lines of table 1: a conflict party's accumulated Goldstein scores *do not Granger cause their daily escalation.* Regardless of which variant of Richardson model from section 2 one chooses, however, such causality is a clear implication. We thus conclude that the data on the conflict at hand are not consistent with Richardson's approach.

3.2. Monte Carlo simulation of the three-agent case

We now turn to our extended version of the Richardson model presented in sub-section 2.7. This model is very hard to solve analytically, particularly so because its Jacobian is three-dimensional. We therefore resort to numerical experimentation, using the Wolfram Language[13] script shown in appendix 4.

The procedure basically allows the user to input a number of trials and iterations. For each trial, the parameter matrix $\mathcal{A}$ and the vector of natural aggression levels $\hat{a}_i$ are filled using a pseudo-random number generator. The user can specify the boundaries for this number generation, and the software ensures that all a_{iii} are non-negative according to the idea behind equation (20). $\mathcal{S}$ always starts filled with zeroes, but is then updated according to equation (20) for the specified number of iterations. We consider the system to *pseudo-converge* if the sum of all elements in $\dot{\mathcal{S}}$ is lower than some hard-coded value, in our case one tenth of a thousand.

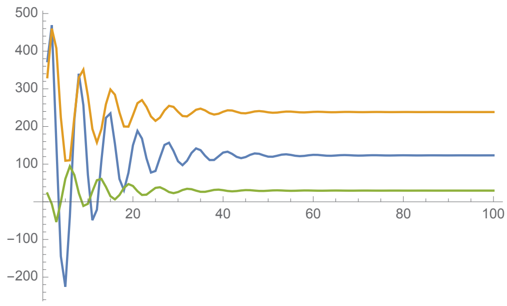
The overall simulation results are interesting. Figure 9 displays selected results for a typical Monte Carlo run with 100 trials using the hard coded parameters from the appendix, plotting the general hostility levels s_{ii} over time. Pseudo-convergence occurs in a minority of cases – 13 % in the example run that figure 9 draws upon –, but it is by no means negligible. The number of repetitions mainly seems to be sufficient for the distincion between unstable and stable scenarios, with the possible exception of situations like the ones in panels (9c) and (9e).

Consistent with our argument about additional dynamic patterns arising in an incrementalist version of the Richardson equations (sub-section 2.2 above), we find several typical patterns of conflict dynamics. Oscillations are a frequent occurence in both stable and unstable simulation trials, but instability is often characterised by monotonic "explosions" like the ones in panels (9b) and (9d) rather than increasing amplitude as in (9f).

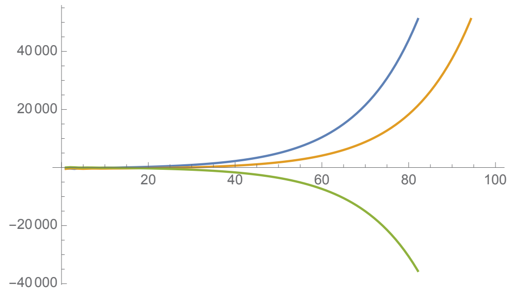
4. Conclusion

The present paper revisits the Richardson equations from both an analytical and an empirical perspective. In so doing, our objective was to see whether this old staple can

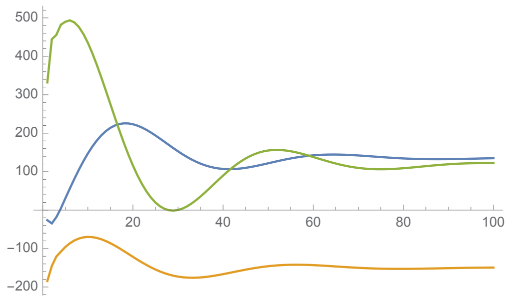
[13]For information regarding this computer language, refer to `https://www.wolfram.com/language/`.



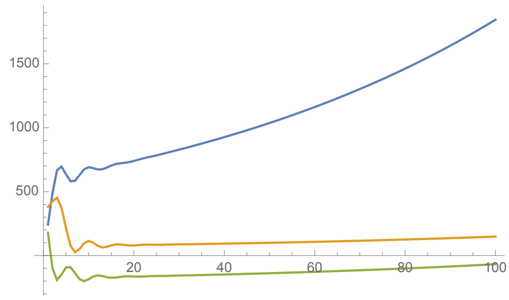
(a) An example of pseudo-convergence



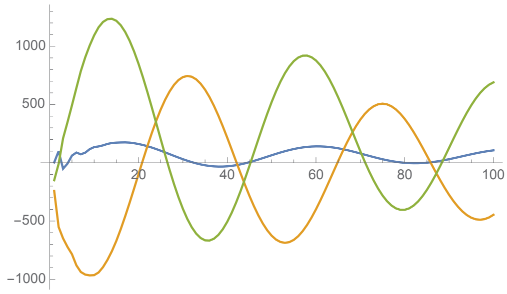
(b) Failure to converge



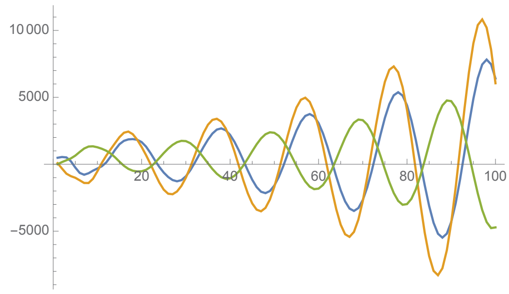
(c) A “near miss”?



(d) Non-convergence with an “outlier”



(e) Stable vortex



(f) Unstable vortex

Figure 9: Typical results for converging and non-converging simulation trials

be brought back from the world of teaching (where it serves as an example for solving systems of differential equations) into modern research on conflict dynamics.

In the analytical part, we find that the dynamic properties of the model can be improved upon by letting a party's level of aggression depend on the *change* in the other party's aggro level ("escalation") rather than on the stock variable. We also note that the probabilistic nature of conflict, which has figured prominently in the military literature since von Clausewitz (1873) made friction a core element of his theory, and which is also reflected by the modern vision of "hybrid" warfare, can be integrated into the Richardson model.

We discuss extensions of the Richardson equations for three-agent scenarios. Relaxing the non-negativity constraint on parameters allows for additional intersting conflict dynamics in the Richardson model. However, we find that a psychological interpretation requires a systematic revision of the Richardson model and an increase in its dimensionality. Using Monte Carlo simulations, we suggest that the likelihood of convergence in this extended framework is rather slim, casting doubt on the appropriateness of the extended Richardson model.

In the empirical section, our main findings are negative as well. This conclusion could arise in at least two ways: first, the Goldstein scores from databases like GDELT may be an inappropriate measure of aggressiveness levels and escalation, and second, the Richardson model may not fit the situation at hand. Given the successful empirical work using Goldstein scores (Goldstein and Pevehouse, 1999, e.g.,), we tend to favour the latter explanation.

Recall that Richardson (1919, 1960a,b) explicitly developed his equation to model *arms races*. It was Boulding (1962) who added a psychological interpretation, modelling how one party's aggression level depends on the *perceived* aggression by the other party, and vice versa. Our empirical results cast doubt on this particular application of the B-R equations. In particular, we raised two objections:

1. The model does not fit the pattern of escalation and de-escalation in a shooting war. One problem is that the grouping of actions into (military) operations and the concomitant constraints – in our example: the Monsoon – is not captured.
2. Any "regression to the mean" of the state variable would imply a causal link between this state and later (de-)escalation, which we fail to find in our time series of daily sums of Goldstein scores.

This criticism need not impinge on using the Richardson model as it was originally designed, or for applications that closely resemble arms races. The treatment of scandals by the media may serve as an example.

A. Program listing

```
params = Array[a, {3, 3, 3}]
states = Array[x, {3, 3}]
changes = Array[y, {3, 3}]
naturals = Array[n, 3]
maxparam = 1
minparam = -1
maxturns = 100
attempts = 100
att = 1;
results1 = {};
results12 = {};
results13 = {};
results2 = {};
results3 = {};
nats = {};
pars = {};
stabruns = 0;
While[att <= attempts,
  (*Initialisierung des Runs*)
  For[i = 1, i < 4, i++,
   {n[i] = RandomReal[{-500, 500}],
    For[j = 1, j < 4, j++,
     {x[i, j] = RandomReal[{-500, 500}],
      y[i, j] = 0,
      For[k = 1, k < 4, k++,
       If[i == k,
        a[i, j, k] = RandomReal[{0, maxparam}],
        a[i, j, k] = RandomReal[{minparam, maxparam}]
        ]
       ]
      }
```

```
    ]
   }
  ];
 AppendTo[pars, params];
 AppendTo[nats, {n[1], n[2], n[3]}];
 (*Beginn des Runs*)

 (*Print["Parameter: ",params];
 Print["Naturals: ",naturals];*)
 turn = 1;
 successes = 0;
 (*Beginn der Timesteps*)
 While[turn <= maxturns,
  {sums = 0;
   For[i = 1, i < 4, i++,
    {For[j = 1, j < 4, j++,
      {tempvar = 0,
       For[k = 1, k < 4, k++,
        If[i == k,
         tempvar = tempvar + a[i, j, k]*(n[i] - x[k, i]),
         tempvar = tempvar + a[i, j, k]*y[k, i]
         ]
        ],
       y[i, j] = tempvar, x[i, j] = x[i, j] + y[i, j]
       }
      ]
     }
    ];
   AppendTo[results1, Part[Part[states, 1], 1]];
   AppendTo[results2, Part[Part[states, 2], 2]];
   AppendTo[results3, Part[Part[states, 3], 3]];
   AppendTo[results12, Part[Part[states, 1], 2]];
   AppendTo[results13, Part[Part[states, 1], 3]];
   (*Stabilitätsprüfung*)
   For[i = 1, i < 4, i++,
    For[j = 1, j < 4, j++,
     sums = sums + Abs[y[i, j]]
     ]];
```

```
    (*Stabilitätskriterium*)

    If[sums < 0.0001, {successes++;
      Print["Pseudo-Stabilität in Run ", att, " zum Zeitpunkt ",
       turn]}]
    };
   turn++];
  (*Print["Run: ",  att , ", Stabile Turns: ",  successes];*)
  (*If[
  successes>0,{stabruns++;Print["Pseudo-Stabilität in Run ", att,
  " zum Zeitpunkt ", turn]}];*)
  att++];
stabruns
(*ListPlot[results1];
ListPlot[results2];
ListPlot[results3];*)
```

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