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Dynamics of Military Conflict: an Economics Perspective

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Zusammenfassung/ Abstract

Using examples for each type of model, we consider dynamic games, differential games, and simulation as alternative ways of extending the standard static economic model of conflict to study patterns of conflict dynamics. It turns out that computational requirements and theoretical difficulties impose tight limits on what can be achieved using the first two approaches. In particular, we are unable to study dynamic military conflict as a series of "battles" that are resolved individually. A simulation study based on a new model of adaptive, boundedly rational decision making, however, is shown not to be subject to this limitation. Plausible patterns of conflict dynamics emerge, which we can link to both historical conflict and standard tenets of military theory.

JEL-Klassifikation / JEL-Classification: C72, D74

Schlagworte / Keywords: Conflict, dynamics, contest success functions, differential games, dynamic games, simulation, emergence of war

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1. Introduction

The rise of game theory in the quantitative social sciences has made considerable potential available for the scientific study of conflict. There exist a large number of applications of game theory (as well as economic theory at large) to particular conflicts and to patterns of conflict in general.[2]

While much of the existing formal analysis is static, conflict is inherently dynamic. The very idea of "seizing the initiative", so prevalent in strategic thought, remains incomprehensible without a dynamic model.

The present paper sets out to explore the avenues open to *economic, quantitative* analysis in the research area of conflict dynamics. We consider dynamic games, differential games, and simulation as alternative ways of extending the standard static economic model of conflict to study patterns of conflict dynamics, and we provide examples for each type of model. All in all, the present paper is probably best described by the artificial term "**surcept**", meaning that *a partial survey of the literature is combined with new research to demonstrate in which direction a particular body of literature might be developed*.

We will start by looking at some static models (section 2), and then move on to discuss the natural way to extend these in a dynamic setting (section 3). Finally, we will move to *simulation studies with boundedly rational agents* (section 4), providing an example for the direction we think quantitative analysis of conflict dynamics can progress the most in.

2. Background: simple static models of conflict

Let us begin by discussing *static* models in conflict economics at some length to provide a background for our main argument. We will not attempt any survey of this literature at all, let alone a thorough one[3] – not only is the relevant body of literature vast, we also need to make just a couple of comments to pave the way for our own analysis in later sections.

2.1. Game theory and its limits for military analysis

Game theory powers most research of strategic interaction in economics, where the term "strategic" does not refer (as it would in military jargon) to the level of decision-making, but just denotes the fact that an individual's payoff does not depend on her actions alone, but on those of her antagonists as well. The "battle of the sexes", represented in *strategic form* in table 1 below provides an excellent example of the reasoning involved.

[2] Myerson (1997) provides a textbook treatment. See Sandler and Atze (2007) for a particular application.

[3] The reader is referred to Garfinkel and Skaperdas (2007) for a recent review.

Table 1: Example – the "battle of the sexes"

		Joan	
		Math	Econ
John	Math	2 , 1	0 , 0
	Econ	0 , 0	1 , 2

Joan and John must independently and without any means of communication decide whether to attend one or the other of two concurrent classes, labeled "Math" and "Econ" for this example. As the two young people are in love, they long to be together, and so abhor any outcome where they are not (represented by an ordinal pay-off of zero – the first number in a cell is John's payoff when that cell is reached, the second is Joan's). Both find it better to be in the same spot (pay-off of 1), but better still if they also attend a lecture that they find interesting. John happens to love Math, while Joan, bless her, much prefers Econ (represented by an ordinal pay-off of two).

Note that this very basic game combines an element of co-operation (both long to be in the same place) as well as an element of conflict (they disagree about where to meet). It therefore exemplifies the mixed motives games that are very much in the focus of modern game theory and also capture the type of strategic situation confronting military decision-makers in modern asymmetric conflicts much better than classical zero-sum models do.[4]

Economics takes its idea of *equilibrium* from Newtonian physics: A state of equilibrium obtains if there are no endogenous forces leading out of it. As the forces driving change in economic models are decisions, an economic equilibrium obtains if no-one has any incentive to revise their decision, given the acts (or decision functionals) of all others. In the example of table 1, (Math,Math) and (Econ,Econ) are obvious (pure strategy Nash) equilibria. There also exists a third equilibrium in mixed strategies, where both John and Joan randomize over their pure choices.

In these game theory models, all options of all players are specified explicitly. In a military setting, for example, we would have to describe detailed options such as placing the main effort of attack to the West of the River versus attacking

[4]Such models, where the best outcome for Blue is the worst for Gold, Blue's second best outcome the second worst for Gold and so forth, were in the center of attention during the founding years of game theory. This comes as little surprise given that the seminal monograph on game theory Neumann and Morgenstern (1944) was published while its authors were busy at RAND analyzing how to beat the Huns. Zero-sum games are indeed representative of much of traditional tactics in military science. However, the permanence of lessons derived from it – such as the general rule that the J2 should base her assessment of the enemy's intention on the worst possible option – in modern tactics does not appear to be altogether justified (Beckmann, 2007).

through the forest in the West, and so forth. There are, however, deep questions concerning what we might hope to learn from such an exercise. There is no doubt that we can use game theory models to apply structure *ex post*, that is to explain singular events in a hermeneutic way. Game theory can thus improve on the usual application of military history, which still lacks a solid theoretical foundation.

Economics, however, typically looks for prognosis. Its blueprint for scientific advance is, again, modeled on the natural sciences, where hypotheses are formulated *ex ante* and then confronted with the data. Given the differences in human motivation, the boundaries of human intelligence, and the vagaries of the human mind, it would be hopeless to aim for an explanation of actual behavior for every single individual. Rather than that, economics seeks to explain *averages*, relating rates of change in decision variables to rates of change in constraints (Pies and Hielscher, 2013, p. 4).

By the same token, game theory cannot provide an underpinning for prognosis on a small scale in a military application. Whenever confronted with a single instance of a particular game, game theory has but limited normative power. Yet, this is precisely the setting that the usual student of tactics, the battalion commander in a conflict, has to consider, pitting his will against the enemy commander's. The divisional commander, on the other hand, can look at the platoon level with the help of game theory, assuming that the individual deviations from the prediction will cancel out. We may conclude that game theory is useful in *providing structure for strategic thought*, and for a *prognosis of events at much lower echelons* than the one whose vantage point we assume.[5]

2.2. The production technology of conflict: contest success functions

Many applications of game theory to the analysis of conflict lack a rich specification of the various options available to the parties. Instead, strategic choices are starkly represented by a continuous variable f_i denoting the fighting effort of party i. If we take this route, we require a *conflict technology* $\pi_i = \pi_i(f_1, \ldots, f_n)$ mapping the fighting efforts of the n contestants into payoffs π. One part of this mapping is the *contest success function*, which in turn maps the fs into the probability of victory p.[6]

The "classical" contest success function shown in eq. (1) below was pioneered in the rent-seeking literature (Tullock, 1987).

$$p_i = \frac{f_i^m}{\sum_{j=1}^{n} f_j^m} \tag{1}$$

[5]This point is essentially made by Clausewitz: "as in all practical arts, the function of theory is to educate the practical man, to train his judgment, rather than to assist him directly in the performance of his duties" (von Clausewitz, 1873, p. 361).

[6]With risk-neutral opponents, this is obviously equivalent to winning a share p of the contested prize.

where the probability of i winning the prize is basically equal to the ratio of i's effort to the overall effort expended on fighting. m is a parameter designed to reflect economies of scale in fighting.

As an example, consider a winner-takes-all scenario where the successful party receives all the resources not expended on fighting. Let all the risk-neutral contestants be identical, assume $m = 1$ and normalize their endowments to unity. The expected payoff of any contestant i is then given by

$$\pi_i = \frac{f_i}{\sum_{j=1}^{n} f_j} \sum_{j=1}^{n} (1 - f_j) \tag{2}$$

With Nash behavior, we differentiate with respect to f_i and simplifying, we obtain the first-order condition for an interior solution[7]

which we then solve for f_i to obtain reaction functions

$$f_i^* = \sqrt{n \sum_{j \neq i} f_j} - \sum_{j \neq i} f_j$$

for all $i \in \{1 \dots n\}$. In an interior symmetric Nash equilibrium, we have $f_i^* = \frac{n-1}{n}$ with the usual conclusions: (a) warfare is a negative sum game, (b) additional competition will lead to harder fighting ($\lim\limits_{n \to \infty} \frac{n-1}{n} = 1$), and by the same token, the deadweight loss of fighting (or legal efforts, or lobbying) increases in n.

Other contest success functions have been explored in the literature, most notably the logistic type, where the probability of victory depends on the *difference* rather than the ratio of efforts expended (Hirshleifer, 2001).

2.3. Lessons for further discussion

One major lesson to be learned from the economic literature on static conflict is that the same formal model can typically be applied to a number of different topics in conflict theory, often both violent and non-violent. For example, the analysis expounded in sub-section 2.2 was applied to lobbying (Tullock, 1987) as well as to arms races (Faeron, 2011). Although claims to have boiled conflict down to the prisoners' dilemma, the stag game (Skyrms, 2004) or the volunteer's dilemma (Diekmann, 1985) are overstated, a small number of game types still

[7]We cannot rule out the possibility of a corner solution *a priori* for this class of models. Especially in the case of asymmetries, certain parties – such as -weak contestants with limited access to resources – may optimally want to expend more than their available resources on fighting. For an extended discussion of this point, see Hirshleifer (2001).

appears to cover most applications, and this suggest *looking for a typology*[8] in dynamic games as well.

Secondly, it appears natural to hope for a *direct extension of static models* to guide research on conflict dynamics. That is, it seems natural to try and extend the well-known static models in conflict economics sketched above by adding a time dimension, distinguishing different periods of time and sequences of moves and counter-moves. We will provide examples of this approach in the following section 3, one of them entirely new, and argue that the limits of available mathematical theory to serve as an underpinning frustrate all hopes for this "natural" approach.

3. Dynamic extensions of standard models: examples and critique

Isaacs (1965) used the term "differential games" for all game theoretic efforts that contain a time index. Modern usage, however, follows Isaacs' earlier practice of restricting this term to multi-agent strategic control problems in continuous time, while the discrete-time version, which typically adds asymmetric information, is generally known as "dynamic games" (Isaacs, 1954). These two make up the avenues of research under scrutiny, and we will consider an example for each in turn.

3.1. The emergence of war in dynamic games

Figure 1 on page 7 shows a modified version of a dynamic game introduced by Gartzke et al. (2003), which serves well to illustrate the main point raised above. A resource valued at unity is in dispute among two countries, A and B. While B's military strength is common knowledge, A can by any of two types: strong or weak.

In a typical analysis, we would want to characterize the equilibria of this game and use these results to discuss the emergence of violence (i.e., war) in this framework. Gartzke et al. (2003), for example, use their version of the model to show that war does not arise because of any (known) cost of war, such as trade benefits, but rather through information asymmetries. For our own purposes, however, it is sufficient to demonstrate how the analysis of the above model would typically proceed.

(Step 1) Start at the final sub-games and roll back. Note that the maximum demand d^* that leaves a strong A just indifferent between fighting back or not is $d^* = 1 + c - a_s$. Likewise $d^{**} = 1 + c - a_w$ for the weak A. h cancels out. $d^{**} > d^*$.

[8] The seminal typology for static games was presented by Rapoport and Gordon (1976).

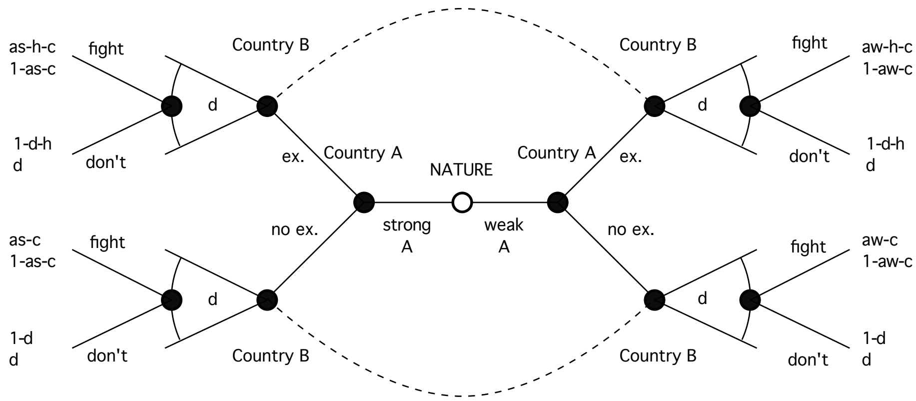

Figure 1: Extensive form for a dynamic game of conflict

(Step 2) Let ϕ (ψ) be the ex post probability B accords to the fact that A is strong, having observed an exercise (no exercise). Observe that it is always optimal to demand more if your opponent caves in. Therefore, A has three options at the upper information set:

1. Fight both types for an expected payoff of $1 - a_w - c - \phi(a_s - a_w)$.
2. Fight only the strong type for an expected payoff of $1 + (1 - 2\phi)c - a_w - \phi(a_s - a_w)$.
3. Fight none and obtain $d^* = 1 + c - a_s$ for sure.

Note the first alternative is dominated by the second as $\phi \leq 1$.

(Step 3) Argue likewise for B's lower information set.

(Step 4) Observe that step 2 implies that a weak A never has to fight. Therefore, this type lacks any incentive to demonstrate strength and we have $\text{prob}(h|w) = 0$.

(Step 5) Let us *construct* a mixed-strategy equilibrium. In this, B would be indifferent between his remaining pure strategies. From step 2, we know that this implies $1 + (1 - 2\phi)c - a_w - \phi(a_s - a_w) = 1 + c - a_s$. From Bayes' Law, we also have

$$\phi = \frac{\text{prob}(h|s)p}{\text{prob}(h|s)p + \text{prob}(h|w)(1 - p)}$$

Using step 4, we can solve the two equations to find

$$\text{prob}(h|s) = \frac{(1 - p)(a_s - a_w)}{2cp} \tag{3}$$

This establishes our equilibrium. The main conclusions from this analysis are that

- as $c\uparrow\Rightarrow d\uparrow$, war cannot be avoided by increasing its cost symmetrically.[9]
- Rather, war arises through *asymmetric* or incomplete *information*.
- From equation (3), we see that the model predicts a strong type to demonstrate the more,
    1. the higher the capability differential $a_s - a_w$,
    2. the lower its ex ante probability p, and
    3. the lower the cost of fighting c.

The main problems with this approach are that such dynamic games of incomplete information do not scale well and may be fraught with multiple equilibria. Regarding the latter point, we need to point out that the model used above is essentially a development of *Quiche* (Cho and Kreps, 1987), which is known to be quite sensitive to assumptions regarding payoffs (Binmore, 1991). The former problem, however, is of larger relevance for the study of practical problems. From a very early point onwards (Isaacs, 1965) it has been well recognized that piling on additional complexities typical of real conflict dynamics tend to increase the tedium of finding the (set of) solutions.

It remains to demonstrate, by way of example, that the same is also true of differential games. Even more than in the case of dynamic games, the problem is not just that finding a solution turns out to be laborious, and that there may be many potential equilibria, but that the apparatus of mathematics in its present state does not afford general solutions. The move from pure conflict to mixed motive games – which are more appropriate for the analysis of post Cold War, asymmetric scenarios – exacerbates this situation.

3.2. A simple differential game model of military conflict

Differential game approaches to modeling conflict are not new. They are an extension to optimal control theory which received some great attention in the time after World War II. Researchers in the USSR (Pontryagin, 1962) and the US (Bellman, 1957) independently looked for dynamic programming approaches to classical calculus of variations problems.

The pioneering application of differential equations models to *military* conflict is the famous Lanchester (1916) model where agents do not optimise. Early applications of differential games with optimising agents on military conflicts mainly focused on how vehicles trying to collide with a rationally evading vehicle

[9]The standard model of resource conflict developed by Skaperdas (2006) also predicts settlement in the absence of inconsistent expectations, commitment problems, and malevolence. Note, however, that the classical liberal hypothesis that trade reduces (violent) conflict receives some empirical support summarized in the textbook by Anderton and Carter (2009). Resolving this debate need not concern us here.

would, or could, achieve contact or minimize distance in a given timeframe.[10] This kind of research would easily fit into a constant-sum framework and was well-suited for application.[11]

The rise of mixed-motive games (Schelling, 1960) prompted an application of differential games to wider strategic questions. Most of the papers on *military* problems, however, concerned arms races – on this, see Deger and Sen (1984); Chang et al. (1996); Lee (2007) –, and technical problems often prevented full solutions to be given. Applications to non-military conflict were relatively more numerous,[12] which we attribute to the the fact that it is easier to avoid non-linear formulations in the latter case (see our discussion in subsection 3.2.3 below). Still, neither of the two approaches can be called a "hot" topic of research in economics, although neither the importance of military conflict nor its prevalance seem to have subsided since the 1970s.

We now introduce a new differantial game model of dynamic conflict that builds on the discussion in sub-section 2.2. We will then use this model to identify the issues preventing the differential game treatment of military conflict from going forward.

3.2.1. Model setup

We analyze a dynamic conflict of known length T. There are two parties to the conflict, an attacker called a and a defender called d, who commit resources $x(t)$ and $y(t)$ out of a common pool $r(t)$ to the conflict at each point of time t. These resource commitments build up force levels $a(t)$ and $d(t)$ according to the differential equations

$$\dot{d}(t) = x(t) - \beta_d d(t) \tag{4}$$

$$\dot{a}(t) = y(t) - \beta_a a(t) \tag{5}$$

where the β_is denote the depreciation rates of the respective force levels. The initial force levels at $t = 0$ are assumed to be known

$$d(0) = d_0 \tag{6}$$

$$a(0) = a_0 \tag{7}$$

Upon termination of the conflict, the remaining resources $r(T)$ are distributed among the parties according to the csf (1). The model is therefore a straightforward extension of the simple static model discussed in subsection 2.2. In

[10] Also known as *pursuit-evasion games* - see Isaacs (1965) for example.

[11] A notable systematic review and extension of these zero-sum problems is Taylor (1970, 1972).

[12] "Relatively" is the term in point. The application of differential games to the study of non-military conflict, for example in resource economics (Clark, 1979), still constitutes a small body of literature.

addition, however, we assume that accumulated forces provide a constant unit benefit of κ, which we interpret as the benefits of power conferred by being in possession of a "fleet in being".

The objective functionals of the parties are (with zero discounting)

$$J_d = \int_0^T \kappa d(t) - x(t)^2 \mathrm{d}t + \sigma \frac{d(T)r(T)}{a(T)+d(T)} \tag{8}$$

$$J_a = \int_0^T \kappa a(t) - y(t)^2 \mathrm{d}t + \sigma \frac{a(T)r(T)}{a(T)+d(T)} \tag{9}$$

while the resource stock depletes according to the kinematic equation

$$\dot{r}(t) = -\chi x(t) - \gamma y(t) \tag{10}$$

and the initial level of the resource is assumed to be known

$$r(0) = r_0 \tag{11}$$

We have to impose the additional constraints that the overall resources can never be fully depleted and that force levels be non-negative.

3.2.2. Analysis

The situation above is said to be a differential game in the *Bolza form* since its objective functionals include additional salvage values for the resource $r(t)$ after T is reached. The Hamiltonians for this problem can be written as follows:

$$H_d = \kappa d(t) - x(t)^2 + \delta(t)(x(t) - \beta_d d(t)) + \rho_d(t)(-\chi x(t) - \gamma y(t)) \tag{12}$$

$$H_a = \kappa a(t) - y(t)^2 + \alpha(t)(y(t) - \beta_a a(t)) + \rho_a(t)(-\chi x(t) - \gamma y(t)) \tag{13}$$

By using *Pontryagin's Maximum Principle* (Pontryagin, 1962) we can find the optimal controls $x(t)^*$ and $y(t)^*$.[13] Inserting them into the state equations (4,5,10) gives us the differential equations for the optimal trajectories $d(t), a(t), r(t)$ which combined with the adjoint equations $\dot{\delta}(t), \dot{\alpha}(t), \dot{\rho}_d(t), \dot{\rho}_a(t)$ result in the

[13] All computations can be found in the appendix.

following system of partial differential equations (PDE):

$$\dot{d}(t) = -\beta_d d(t) + \frac{1}{2}(\delta(t) - \chi\rho_d(t)) \tag{14}$$

$$\dot{a}(t) = -\beta_a a(t) + \frac{1}{2}(\alpha(t) - \gamma\rho_a(t)) \tag{15}$$

$$\dot{r}(t) = -\frac{1}{2}\chi(\delta(t) - \chi\rho_d(t)) - \frac{1}{2}\gamma(\alpha(t) - \gamma\rho_a(t)) \tag{16}$$

$$\dot{\delta}(t) = -\kappa + \beta_d\delta(t) \tag{17}$$

$$\dot{\alpha}(t) = -\kappa + \beta_a\alpha(t) \tag{18}$$

$$\dot{\rho_d}(t) = 0 \tag{19}$$

$$\dot{\rho_a}(t) = 0 \tag{20}$$

The resolution of conflict at time T and the resulting allocation of the remaining common pool resource $r(T)$ is captured by the transversality conditions. By differentiating the salvage values with respect to $r(T)$ we find the ones for the $\rho_i(T)$s:

$$\rho_a(T) = \sigma\frac{a(T)}{a(T) + d(T)} \tag{21}$$

$$\rho_d(T) = \sigma\frac{d(T)}{a(T) + d(T)} \tag{22}$$

Since the accumulated forces are of no further value for the objective after the conflict terminates, the corresponding transversality conditions are of the form

$$\alpha(T)a(T) = 0 \tag{23}$$

$$\delta(T)d(T) = 0 \tag{24}$$

This also fixes the $\rho_i(t)$s at a constant level. As a result, the above system is simple enough to afford a closed-form solution (see the appendix). However, the main results of this model are best demonstrated by way of a numerical example illustrating the equilibrium paths of key variables. We find that unless the rate of depreciation is too high, military build-up will be initially high and fall over time (see figure 2 for a typical example; lower depreciation levels lead to higher initial effort, *ceteris paribus*).

The economic intuition behind this result is that a rapid build-up allows the players to derive more consumption / power benefit from any given effort, *c.p.*. As a result, optimal force levels can actually *fall* at the end of the planning horizon, i.e. a short time before the terminal "hot" phase of the conflict (as shown in figure 3).

Figure 4 plots the time paths of $r(t)$ for this example. We see that the common pool depletes over time, which exemplifies the wasteful nature of conflict. In this scenario, lower depreciation levels lead to less exploitation of the resource,

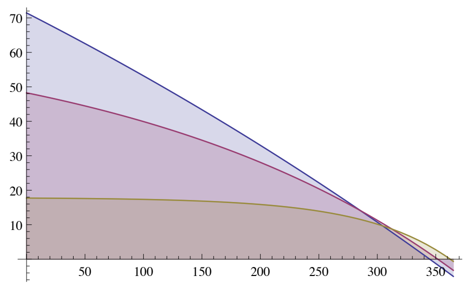

Figure 2: Defender's optimal effort level

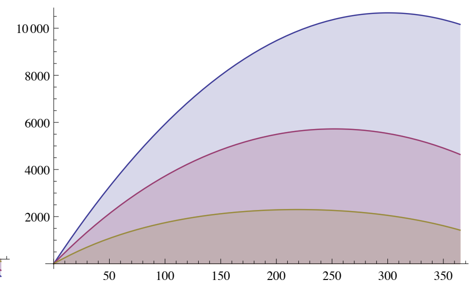

Figure 3: Defender's optimal force level

c.p.. Yet, in models of this type it will never deplete completely because the shadow prices $\rho_i(T)$ of the resource at the termination of the game are positive (the positive gain from seizing parts of the resource).

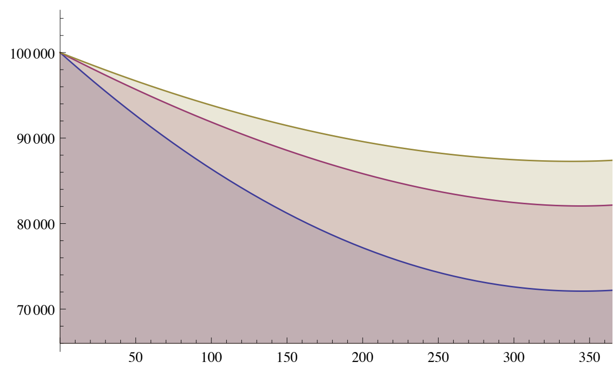

Figure 4: Resource levels over time

3.2.3. Discussion

At first blush, the model discussed in this section appears to be a straightforward extension of the standard static model from sub-section 2.2. But note that while the build-up of forces – and the concomitant consumption of "fleets in being" – is dynamic, *the resolution of the conflict over the common pool resource is not*. We still model this as occurring in a single cataclysmic confrontation at the end of the dynamic game. Furthermore, the common pool is unproductive in our model.

By way of the transversality conditions, the *single* application of the ratio csf at time T determines a constant shadow price $\rho_i(t)$ of the resource, and the only dynamic aspect of the model is how optimally to distribute the effort undertaken in preparation for this conflict over time. This part of the model, in turn, basically involves an isolated trade-off between utility derived from a standing army on the one hand and depreciation on the other. The only strategic dimension at this point is that investment at time τ reduces the common pool available for *both* parties at all times t with $T > t > \tau$.

A satisfactory model would treat dynamic conflict as a series of instantaneous battles in continuous time. However, we were unable to formulate such a model whose system of PDEs could be solved analytically while *retaining a standard csf of either the ratio or logistic type*. These csfs introduce non-linearities into the model, which can lead to problems in optimal control theory (Feichtinger and Hartl, 1986) and are typically avoided in differential games.[14]

We thus end on the horns of a dilemma: either reduce the complexity of the model until it loses some essential features of conflict (e.g., adequate conflict resolution technology or true dynamism) or refrain from providing an analytical solution for the optimum. In fact, our analysis in the preceding sub-section 3.2.2 is fairly typical for applications of differential games to conflict theory in that the main argument is made by way of example – in other words, by *simulating* a solution.

4. A simulation of symmetric conflict between boundedly rational agents

The preceding section 3 makes the point that simulation cannot be avoided if we want to study models of dynamic conflict that include (a) a sequence of "battles" and (b) non-linear conflict success technologies. (A similar argument was proffered regarding the extension of dynamic non-cooperative games.) We now develop an example for such a simulation study. Our object in doing so is not only to address the shortcomings pointed out in the preceding section, but also to introduce a form boundedly rational decision-making that appears helpful in simulation.

4.1. Model structure and software setup

Again, we base the model on the standard ratio contest success function (sub-section 2.2) with two players. In contrast to the differential game discussed previously (sub-section 3.2), the analysis will now be done in discrete time, with one "battle" per period. As a result of this battle at time t, player i will lose an amount

$$l_i(t) = \frac{x_j(t)}{x_i(t) + x_j(t)} r_i(t)$$

[14]Note that there is no general method available for an analytical solution of arbitrary PDEs.

which his opponent, player j, will gain as "booty" (r denotes the players' resources, x their effort levels, and $i, j \in \{A, B\}$ with $i \neq j$). Resources not devoted to fighting $(r(t) - x(t))$ will be used in production, with a productivity factor $\phi_i > 1$ which we allow to differ between players.

Both agents are boundedly rational as we assume them to exhibit a form of adaptive expectations as well as optimize in an *incomplete myopic "optimization"* way. In particular,

1. effort in the first period $x_i(1)$ for $i \in \{A, B\}$ is exogenously given and may differ between players. This captures the *operational readiness* for war at the beginning of the conflict,
2. in choosing their efforts at time $t > 1$, players assume that their opponent will continue as in the previous period,
3. each player will compute the *static* best response $x_i^*(t) = f(r_i(t), r_j(t), x_j(t-1))$ to his opponent's choice in the last period,
4. and then act to close $100\mu_i$ percent of the difference between his own last-period choice and the current "best response". We therefore let $x_i(t) = \mu_i(x_i^*(t) - x_i(t-1)) + x_i(t-1)$ and conceive of μ_i as reflecting the *reaction speed* of the players. Again, players can have different reaction speeds.

A player is said to be defeated whenever it has zero resources, which also ends the conflict in our simulation.

We use the software package STELLA[15] to implement the model. Figure 5 below shows how a player (A) is modeled in STELLA's graphical model editor. The central variable is A's stock of resources, which is augmented by inflows – production and "booty" from military exploits – and reduced by losses – the other party's "booty" and military expenditure – periodically. The latter is computed in the decision diamond and depends on the resources as well as the past effort levels of both parties.

4.2. Results

We used the model for a series of simulation runs with different parameters, in particular, we considered alternative configurations of productivities, operational readiness, and decision speeds. We report on two typical results: First, we consider the case where player B has a higher resource endowment as well as higher productivity, while player A's initial readiness is higher than B's. Figure 6 on page 16 plots the resource levels over time. The plot illustrates that A typically enjoys some initial successes, but that after some time the higher productivity and greater resources of B turn the scales. B's economy actually

[15]STELLA is computer software provided by ISEE systems – see `http://www.iseesystems.com/softwares/Education/StellaSoftware.aspx/` – for systems modeling and simulating. Version 9.3 of STELLA was used because the most recent version – STELLA 10 – dropped support for the decision node construct.

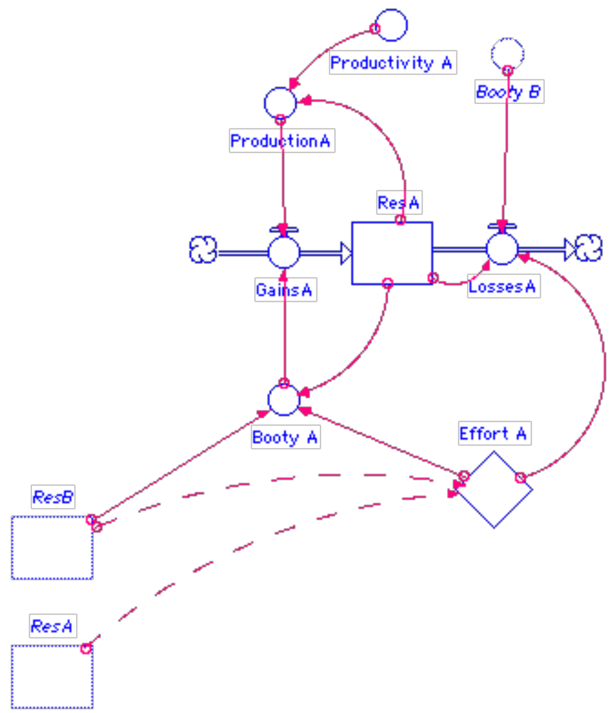

Figure 5: Player A as modelled in STELLA

exhibits positive net growth in the final phase of the conflict, which precedes A's defeat. These stylized dynamics bear a distinct resemblance to the history of the Second World War, both in the European and the Pacific theater, and other historical examples are not hard to find.

In our second simulation, both parties to the conflict are exactly the same, except for the fact that A reacts more quickly than B (in the sense outlined above). Again, the typical result is that the quicker player wins, all other things being equal, which is in line with most military teaching (Simpkin, 1998; Frieser, 2005; Leonhard, 1994).

5. Conclusion

These simulation exercises, albeit simple, illustrate how simulation can be employed to extend the basic static model of conflict. We also use them as a framework for a new and (we hope) convincing way to model adaptive, boundedly rational decisions in dynamic conflict. As a result, plausible patterns of conflict dynamics emerge, which we can link to both historical conflict and standard tenets of military theory.

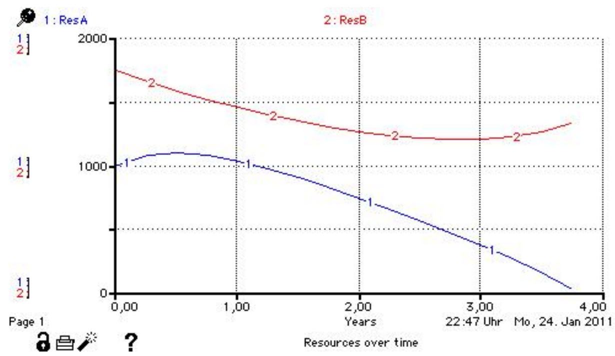

Figure 6: A "World War II" scenario

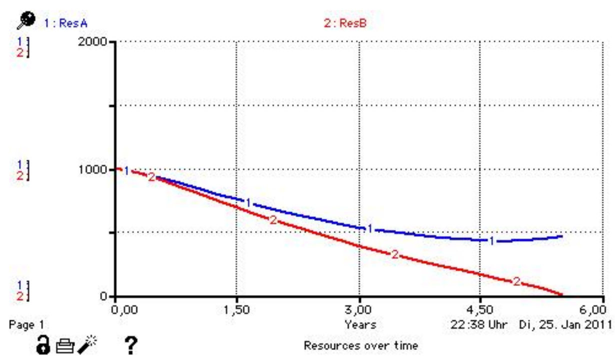

Figure 7: A "Race to the Swift" (Simpkin, 1998) scenario

This compares favorably to the computational effort and theoretical difficulty involved in trying to find analytical solutions and characterize equilibria in both dynamic and differential games. In particular, we can now use a standard ratio contest success function, which cannot be done in an analytical approach unless one is prepared to settle for a model that is not truly dynamic in the sense that (periodic, instantaneous) success in conflict is determined continually as the dynamics unfold.

Appendix: a solution to the differential game from subsection 3.2 using *Mathematica*

(*objective functions*)

$J_1 = \int_0^T \left(\kappa d[t] - x[t]^2\right)\, dt + \frac{\sigma d[T] r[T]}{a[T]+d[T]}$

$J_2 = \int_0^T \left(\kappa a[t] - y[t]^2\right)\, dt + \frac{\sigma a[T] r[T]}{a[T]+d[T]}$

(*hamiltonians*)

$$H_1 = \kappa d[t] - x[t]^2 + (-\beta 1 d[t] + x[t])\delta[t] + (-\chi x[t] - \gamma y[t])\rho 1[t]$$

$$H_2 = \kappa a[t] - y[t]^2 + (-\beta 2 a[t] + y[t])\alpha[t] + (-\chi x[t] - \gamma y[t])\rho 2[t]$$

(*FOCs for optimality*)

$$x[t] \to \frac{1}{2}(\delta 1[t] - \chi\rho 1[t])$$

$$y[t] \to \frac{1}{2}(\alpha 2[t] - \gamma\rho 2[t])$$

(*SOCs*)

$$\frac{\partial H_1}{\partial x[t]} = -2 \leq 0$$

$$\frac{\partial H_2}{\partial y[t]} = -2 \leq 0$$

(*pde system nash closed-loop*)

$$\delta'[t] == -\kappa + \beta 1\delta[t]$$

$$\rho 1'[t] == 0$$

$$\alpha'[t] == -\kappa + \beta 2\alpha[t]$$

$$\rho 2'[t] == 0$$

$$r'[t] == -\frac{1}{2}\chi(\delta[t] - \chi\rho 1[t]) - \frac{1}{2}\gamma(\alpha[t] - \gamma\rho 2[t])$$

$$d'[t] == -\beta 1 d[t] + \frac{1}{2}(\delta[t] - \chi\rho 1[t])$$

$$a'[t] == -\beta 2 a[t] + \frac{1}{2}(\alpha[t] - \gamma\rho 2[t])$$

(*analytic solution*)

$$a \to \frac{e^{-t\beta 2}\left(e^{-t\beta 2}+e^{t\beta 2}\right)\kappa}{4\beta 2^2}+\frac{e^{-2t\beta 2}\left(-1+e^{2t\beta 2}\right)\kappa}{4\beta 2^2}+e^{-t\beta 2}c_1+\frac{e^{-t\beta 2}\left(-1+e^{2t\beta 2}\right)c_4}{4\beta 2}-\frac{e^{-t\beta 2}\left(-1+e^{t\beta 2}\right)\gamma c_7}{2\beta 2}$$

$$d \to \frac{e^{-t\beta 1}\left(e^{-t\beta 1}+e^{t\beta 1}\right)\kappa}{4\beta 1^2}+\frac{e^{-2t\beta 1}\left(-1+e^{2t\beta 1}\right)\kappa}{4\beta 1^2}+e^{-t\beta 1}c_2+\frac{e^{-t\beta 1}\left(-1+e^{2t\beta 1}\right)c_5}{4\beta 1}-\frac{e^{-t\beta 1}\left(-1+e^{t\beta 1}\right)\chi c_6}{2\beta 1}$$

$$r \to -\frac{e^{-t\beta 2}\left(-1+e^{t\beta 2}\right)\gamma\kappa}{2\beta 2^2}-\frac{e^{-t\beta 1}\left(-1+e^{t\beta 1}\right)\kappa\chi}{2\beta 1^2}+\frac{1}{2}\kappa\left(-\frac{e^{-t\beta 2}\gamma}{\beta 2^2}-\frac{t\gamma}{\beta 2}-\frac{e^{-t\beta 1}\chi}{\beta 1^2}-\frac{t\chi}{\beta 1}\right)+$$
$$c_3-\frac{\left(-1+e^{t\beta 2}\right)\gamma c_4}{2\beta 2}-\frac{\left(-1+e^{t\beta 1}\right)\chi c_5}{2\beta 1}+\frac{1}{2}t\chi^2 c_6+\frac{1}{2}t\gamma^2 c_7$$

$$\alpha \to \frac{\kappa}{\beta 2} + e^{t\beta 2}c_4$$

$$\delta \to \frac{\kappa}{\beta 1} + e^{t\beta 1}c_5$$

$$\rho 1 \to c_6$$

$$\rho 2 \to c_7$$

$$c_1 \to a(0) - \frac{\kappa}{2\beta 2^2}$$

$$c_2 \to d(0) - \frac{\kappa}{2\beta 1^2}$$

$$c_3 \to \frac{\beta 1^2\gamma\kappa + \beta 2^2\kappa\chi + 2\beta 1^2\beta 2^2 r(0)}{2\beta 1^2\beta 2^2}$$

$c_4 \to -\frac{\kappa e^{\beta 2(-T)}}{\beta 2}$

$c_5 \to -\frac{\kappa e^{\beta 1(-T)}}{\beta 1}$

$c_6 \to \frac{\sigma d(T)}{a(T)+d(T)}$

$c_7 \to \frac{\sigma a(T)}{a(T)+d(T)}$

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