Sequential competitive location on networks†

Dominik Kress*, Erwin Pesch

Department of Management Information Sciences, University of Siegen, Hölderlinstraße 3, D-57068 Siegen, Germany

Abstract

We present a survey of recent developments in the field of sequential competitive location problems, including the closely related class of voting location problems, i.e. problems of locating resources as the result of a collective election. Our focus is on models where possible locations are not a priori restricted to a finite set of points. Furthermore, we restrict our attention to problems defined on networks. Since a line, i.e. an interval of one-dimensional real space, may be interpreted as a special type of network and because models defined on lines might contain ideas worth adopting in more general network models, we include these models as well, yet without describing them in detail for the sake of brevity.

Keywords: location, competitive location, sequential location, spatial competition, network location

1. Introduction and scope of review

Location problems are concerned with the location of (physical or nonphysical) resources in some given space. Much work in this field has, for example, been devoted to the choice of optimal locations in networks according to some criterion, e.g. the minimization/maximization of the sum of weighted distances to the vertices (median/antimedian problems) or the minimization of the maximum/maximization of the minimum weighted distance to any vertex (center/anticenter problems). *Competitive location models* additionally incorporate the fact that location decisions have been or will be made by independent decision-makers who will subsequently compete with each other, e.g. for market share when we think of locating facilities such as gas stations or supermarkets. The location space under consideration does not necessarily need to be of geographical nature: Political parties, for example, are concerned with locating in issue spaces; products may be positioned in characteristics spaces.

The study of competitive location problems is rooted in the work of Hotelling (1929), who studied the location choice and pricing decision of two competitors on a finite line with uniformly spread consumers. A large number of publications have been devoted to this

*Corresponding author, phone: +49 271 740 3402, fax: +49 271 740 2940

Email addresses: dominik.kress@uni-siegen.de (Dominik Kress), erwin.pesch@uni-siegen.de (Erwin Pesch)

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field of research since then. Thus, several efforts in classifying and reviewing competitive location models have been made. Table 1 lists some of the resulting publications.

Table 1: Competitive location problems - reviews and classifications.

Authors and year	Class	Type
Dasci (2011)	competitive location	classical contributions, review
Plastria (2001)	static competitive location	taxonomy and review
Eiselt & Laporte (1996)	sequential competitive location	taxonomy and review
Drezner (1995a)	competitive location in the plane	review
Serra & ReVelle (1995)	competitive location in discrete space	review
Eiselt et al. (1993)	competitive location	taxonomy and bibliography
Eiselt (1993)	competitive location	taxonomy
Eiselt & Laporte (1989)	competitive location	taxonomy and review
Friesz et al. (1988)	competitive location in networks	taxonomy and review
Graitson (1982)	competitive location on a line	review

Competitive location problems can be classified with respect to multiple components (see, for example, Eiselt, 1993; Eiselt & Laporte, 1989, 1996; Eiselt et al., 1993; Friesz et al., 1988; Hamacher & Nickel, 1998; Plastria, 2001). Most important, the *representation of the underlying location space* traditionally gives rise to three classes: d-dimensional real space, network and discrete space. Distances need to be calculated by some metric in each of these classes. We follow ReVelle & Eiselt (2005) in differentiating only between d-dimensional real space and network location problems, each of which further being subdivided into continuous and discrete problems (see Figure 1). A *discrete* problem arises, when the set of candidate locations is assumed to be finite and known a priori. In a *continuous* problem, any point of the network or the d-dimensional space is a potential location site. By identifying finite dominating sets - finite sets of points to which at least one of the optimal solutions must belong - we are able to transform some special classes of continuous location problems into discrete problem classes a posteriori (Hooker et al., 1991). Moreover, as discrete sets of potential facility sites may easily become very large, one may consider treating those sets as continuous entities (see, for instance, Dasci & Laporte, 2005).

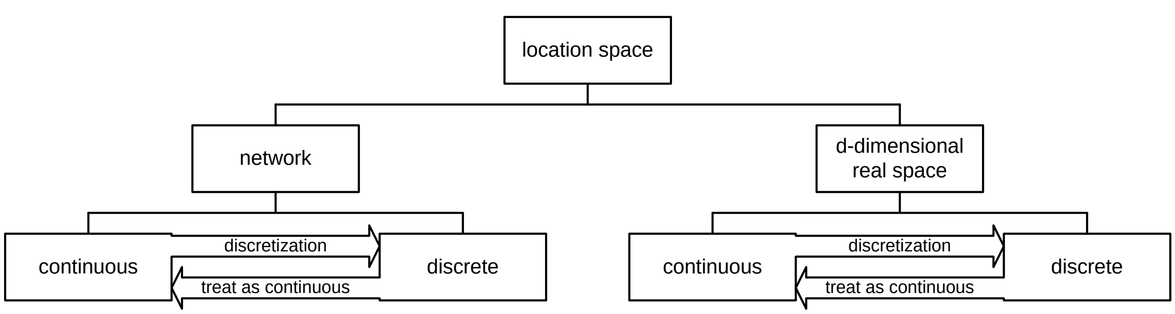

Figure 1: Location spaces.

Note that there is an overlap of d-dimensional real spaces and networks. Hotelling's line, for instance, can be interpreted as a network with two vertices and a connecting edge or, alternatively, as an interval of $\mathbb{R}^1$. Other models are somewhere in between network and d-dimensional models. Suárez-Vega et al. (2010), for example, consider a "buffer" around a network to represent the space of potential locations.

Other fundamental categories of competitive location theory are related to game theoretic aspects. Competition itself may be static (present and fixed), competitors may enter in a simultaneous or sequential fashion, or we can think of dynamic competition, i.e. players who repeatedly reoptimize their locations (see Figure 2). *Sequential locational competition*, dating back to Hay (1976) and Prescott & Visscher (1977), is characterized by two types of players: leaders, who choose locations at given instants, anticipating the subsequent actions of later entrants, and followers, who make their location decisions based on the past decisions of the leaders. The solution concept generally employed in sequential location problems is the Stackelberg equilibrium (von Stackelberg, 1934): Assuming rational players, the location of each player is determined by backward induction. *Simultaneous locational games* (as the one of Hotelling, 1929), in contrast, usually use the concept of a Nash equilibrium. Here we are seeking situations where no player unilaterally has the incentive to relocate.

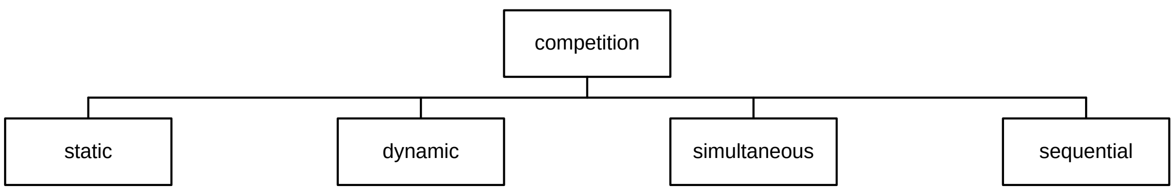

Figure 2: Competition.

Note that the *number of players* may be exogenously given or determined endogenously, e.g. by incorporating fixed location cost. The same holds for the *sequence* of location and the *numbers of resources* to be located by each player.

It is generally agreed that the work on competitive location problems on (general) networks is rooted in the work of Slater (1975) and Hakimi (1983) (see Eiselt & Laporte, 1996; Smith et al., 2009, for more details). Hakimi (1983) formally introduced the terms $(r|X_p)$-medianoid problem and $(r|p)$-centroid problem for sequential games with one leader (L) and one follower (F) locating p and r facilities, respectively. Note that r and p are arbitrary input parameters. Knowing the p locations of L, denoted by $X_p = (x_1, ..., x_p)$, F faces the problem of optimally locating r facilities (with respect to some objective function): the *$(r|X_p)$-medianoid problem*. We denote a feasible location decision of F by $Y_r = (x_{p+1}, ..., x_{p+r})$ and an optimal location decision by $Y_r^* = (x_{p+1}^*, ..., x_{p+r}^*)$. L's problem, the *$(r|p)$-centroid problem*, is to locate p facilities, anticipating F's subsequent behavior. An optimal solution to this latter problem is denoted by $X_p^* = (x_1^*, ..., x_p^*)$. Note that, differing from other authors as Spoerhase & Wirth (2010), we use the terms $(r|X_p)$-medianoid and $(r|p)$-centroid problem in a rather broad sense, subsuming a whole variety of choice rules and player objectives under these terms.

Another category that is related to game theory is the incorporation of pricing in competitive facility or product location models. Traditional spatial pricing policies include *mill pricing* (all customers are charged the same price for the good itself, all transport costs are passed to the customers), *uniform delivered pricing* (all customers of a facility are charged the same price, no matter where they are located at) and *spatial price* *discrimination* (prices for different customers are customer-location-specific) (see, for example, Anderson et al., 1992b; Eiselt & Laporte, 1996; Eiselt et al., 1993; García Pérez et al., 2004). Prices may be set simultaneously to the location decisions or in a separate stage, either sequentially or simultaneously. Alternatively, one can incorporate *parametric prices*. The equilibrium concepts used for combined location price games depend on which of these situations is implemented (Eiselt et al., 1993).

Other ingredients of competitive location models include *characteristics of the targeted group*, as, for example, customers or voters. They may be distributed over the representation of the location space according to some density function or we may consider discrete locations. The type of demand for the resources, that are to be located, may be *deterministic* or *stochastic*, *elastic* or *inelastic* (dependent on or independent of the conditions of its supply). Furthermore, we may take different types of choice rules into consideration: a choice rule is said to be *deterministic* (or *binary*), when the total demand of a customer (voter etc.) is served by a single located resource; it is said to be *probabilistic*, if demand is split over multiple located resources. Probabilistic choice rules include *partially binary* (splitting only over one of the locations of each player) and *proportional* (splitting over all locations) behavior. Note that, at least in the case of a binary choice rule, one has to make assumptions concerning the location of two facilities in the same point of the network (*co-location*). A common assumption in the field of competitive location problems is to break ties in favor of the leader (see, for example, Hakimi, 1990; Hansen & Labbé, 1988; Hansen & Thisse, 1981) or, similarly, not allowing co-location at all (see, for example, Granot et al., 2010; Shiode & Drezner, 2003). Alternatively, ties may be broken equally as, for instance, in Dasci et al. (2002). Both, the existence and nature of equilibria in competitive location models, may vary according to different assumptions concerning co-location: Hakimi (1990) designs the above-mentioned tie breaking rule to "avoid [...] trivial solutions". Similarly, Granot et al. (2010) analyze the effect of allowing or not allowing co-location in their model in detail.

A whole variety of other features may be incorporated into competitive location problems. They include incomplete information: For example, information about marginal costs of production may be asymmetric. Moreover, different types of players pursue different types of objectives: public firms, for instance, maximize domestic welfare, while private firms maximize profits. We may consider multiple products or multiple markets as well.

Eiselt & Laporte presented a survey of sequential competitive location problems in 1996. Various publications in this field have appeared since then. It is the aim of this paper to extend Eiselt & Laporte (1996) by reviewing some of these recent developments. Thus, we are restricting ourselves to detailed descriptions of publications that have appeared after Eiselt & Laporte (1996). Furthermore, we explicitly exclude location decisions in political issue spaces and refer the reader to Osborne (1995) and the references therein. Finally, we restrict our attention to continuous network models as described above. We concentrate on surveying general network models in the traditional sense in detail, while only referencing or listing publications related to $\mathbb{R}^1$ location spaces. We hold this to be reasonable for the sake of brevity while not omitting interesting ideas that might be adopted in general network models. Tables 1 (competitive location problems) and 2 (other fields of location theory) give an overview of several selected review and classification papers that complement this review. We refer the reader to the books by Anderson et al. (1992a), Daskin (1995), Drezner (1995b), Drezner & Hamacher (2002), Eiselt & Marianov (2011), Handler & Mirchandani (1979), Love et al. (1988), Miller et al. (1996) and Mirchandani & Francis (1990) as well.

Table 2: Other selected reviews and classifications.

Authors and year	Class	Type
	Location problems in general	
Smith et al. (2009)	location	historical development
ReVelle & Eiselt (2005)	location	review
Hale & Moberg (2003)	location	review
Avella et al. (1998)	location	state of the art and future trends
Hamacher & Nickel (1998)	location	classification scheme
Domschke & Krispin (1997)	location	review
Brandeau & Chiu (1989)	location	overview
Domschke & Drexl (1985)	location	bibliography
Tansel et al. (1983a,b)	location	review
Francis & Goldstein (1974)	location	bibliography
	Discrete location problems	
ReVelle et al. (2008)	discrete location in special branches	taxonomy and bibliography
Daskin (2008)	discrete location	taxonomy and illustrations
Current et al. (2002)	discrete location	review
McGinnis (1977)	discrete location	review
	Continuous, d-dimensional location problems	
Plastria (2002)	continuous covering location	review
Plastria (1995)	continuous location problems	review
	Location problems with special application or context	
Melo et al. (2009)	facility location and supply chain management	review
Nagy & Salhi (2007)	location and vehicle routing	review
Şahin & Süral (2007)	hierarchical facility location	review
Snyder (2006)	stochastic and robust facility location	review
Klose & Drexl (2005)	facility location for distribution system design	review
Owen & Daskin (1998)	strategic facility location	review
Osborne (1995)	spatial political competition	review
Mesa & Boffey (1996)	extensive facility location in networks	review
Erkut & Neuman (1989)	undesirable facility location	classification scheme and review
Aikens (1985)	facility location for distribution planning	review

Voting location problems (see Hansen et al., 1990, for an introduction), i.e. problems of locating resources as the result of a collective election, are closely related to competitive location problems. Even though the users vote in order to locate resources that can be interpreted to be owned by a single decision maker, the voting process is such that a stable location is generally characterized by the nonexistence of a strong party of users who prefer alternative locations, that may be seen as possible locations of a competitive decision maker. Thus, although the focus of voting location problems is somewhat different, they may even be interpreted as a special case of sequential competitive location problems. There is a strategic element that these problems share (Hansen et al., 1990): Independent decision makers (leaders and followers or voters) influence the solution by making their decisions. Voting location problems are therefore included in this paper.

Another class of closely related location problems is concerned with *coverage* objectives. These types of objectives model situations in which the users seek to have a resource within a reasonable distance. "There is [...] some competitive flavor to such problems in that [...] existing facilities may belong to one company while a second company is trying to extract the maximum profit by locating its own facilities" (Megiddo et al., 1983). An excellent, very recent survey on some classes of coverage problems is Berman et al. (2010). We refer to this review and the references therein for details on coverage problems and survey only indispensable models in this paper.

The remainder of this paper is organized as follows. The notation and definitions used in the review are given in Section 2. Sections 3 and 4 deal with voting location problems. Sections 5 and 6 discuss selected $(r|X_p)$-medianoid and $(r|p)$-centroid problems, respectively. Section 7 is devoted to sequential location problems that cannot be interpreted in the sense of Hakimi's notion as described above. Furthermore, selected models on a line are listed in tables. The sensitivity of sequential competitive location problems with respect to the underlying modeling assumptions is subject of Section 8. The paper ends with a conclusion in Section 9. We will generally use the terms user, customer and voter as well as the terms facility and resource interchangeable. Furthermore, unless otherwise stated, we will assume that the players have the objective of maximizing profit, which, in the absence of prices, becomes equivalent to maximizing market share.

2. Notation and definitions

Unless otherwise stated, the models are described using the notation of Bandelt (1985) throughout the paper (cf. also Bauer et al., 1993). This notation is extended to include new features in the following sections. A *network* $N = (V, E, \lambda)$ consists of a finite set V ($|V| = n$), a finite set E ($|E| = m$) of two-element subsets of V and a mapping $\lambda : E \to \mathbb{R}^+$. The pair (V, E) gives a *graph* in the usual sense (cf. Swamy & Thulasiraman, 1981). The elements v of V are called *vertices* of the network. The elements e of E are the *edges*. Every edge joins two distinct vertices of N. If e is a unique edge joining u and v this is expressed by the shorthand $e = [u, v]$. Unless otherwise stated, we assume that all edges are undirected, hence $[u, v] = [v, u]$. The value $\lambda(e)$ is the *length* of e. We define $\hat{D} := \max_{e \in E} \lambda(e)$. The *points* x of N ($x \in N$) are the elements of the edges (including all vertices). Two points x and y on an edge e ($x, y \in e$) determine a *subedge* $[x, y]$ of e, the length of which is denoted by $\lambda([x, y])$. A *path* $P(x, y)$ joining two points $x \in [u, v]$ and $y \in [w, z]$ is either a subedge or a sequence of edges and (at most two) subedges passing at most once through each point, where $P(x, y)$ contains x and y but no proper connected subset of $P(x, y)$ does. The points x and y are the *end points* of $P(x, y)$. The length of $P(x, y)$ is equal to the sum of the lengths of the edges and subedges. If the length of $P(x, y)$ is minimum among all paths connecting x and y, then $P(x, y)$ is a shortest path; its length is the *distance* $d(x, y)$ between x and y. We define $D(p, Z) := \min\{d(p, z)|z \in Z\}$ for a point $p \in N$ and a set of points $Z \subseteq N$. A *cycle* consists of an edge e joining two vertices u and v and some path $P(u, v) \neq e$ connecting u and v. A network is connected if for any two points x and y there exists a path joining x and y. A connected network without cycles is a *tree network*. A tree network where every vertex is incident to at most two edges is a *chain network*. Unless otherwise stated, we assume that the networks considered in this paper are connected and that there are no loops (edges $[u, u]$) at the vertices.

A *path-decomposition* of a graph $G = (V, E)$ is a sequence $V_1, ..., V_r$ of subsets of the vertex set V, such that

1. $\bigcup_{1 \leq i \leq r} V_i = V$,
2. there exists a V_i, $i \in \{1, ..., r\}$, such that $u \in V_i$ and $v \in V_i$ for all $[u, v] \in E$, and
3. $V_i \cap V_k \subseteq V_j$ holds for all $1 \leq i < j < k \leq r$.

The *pathwidth* of a path-decomposition $V_1, ..., V_r$ is defined as $\max_{1 \leq i \leq r} |V_i| - 1$. The pathwidth of a graph $G = (V, E)$ is the minimum pathwidth over all path-decompositions of G.

Let V' be a subset of the vertex set of N. The network $N' = (V', E', \lambda')$ is the *subnetwork* of N on the vertex set V', if E' is a subset of E such that each edge of E joining u and v belongs to E' if and only if u and v are in V'. The mapping λ' is the restriction of λ to E'.

Almost all considered problems feature a finite number of users located at the vertices of the network N. At each vertex there may be several users or none at all. Unless otherwise stated, their demand is described by a weight function $\pi : V \to \mathbb{R}_0^+$, where π is different from the zero function. For a subnetwork N' of N we denote by $\pi(N')$ or $\pi(V')$ the sum $\sum_{u \in V'} \pi(u)$ where V' is the vertex set of N'.

3. Voting location: Condorcet points and related concepts

The vertex set V of a network $N = (V, E, \lambda)$ with vertex weight function π is partitioned into three sets with respect to any pair x, y of points and a given $\alpha \geq 0$:

$$[x \succ y] := \{u \in V | d(u, x) + \alpha < d(u, y)\}, \tag{1}$$
$$[x \sim y] := \{u \in V || d(u, x) - d(u, y)| \leq \alpha)\}, \tag{2}$$
$$[y \succ x] := \{u \in V | d(u, x) - \alpha > d(u, y)\}. \tag{3}$$

Thus, a customer with deterministic and inelastic demand is indifferent ($\sim$) about two locations, if the difference of their distances to the customer is within the given threshold α (equal for all customers in the network). $\pi([x \succ y])$ denotes the sum of the weights of all those vertices which prefer ($\succ$) x to y.

A well known concept that is based on these definitions, where $\alpha = 0$ and another parameter $0 \leq \beta \leq 1$ is introduced, is the β-*Condorcet point* (Bandelt, 1985). A point x of the network is said to be a β-Condorcet point, if

$$\pi([y \succ x]) + \beta\pi([x \sim y]) \leq 0.5\pi(N) \qquad \forall y \in N. \tag{4}$$

Hence, a β-Condorcet point is any point x, such that no strict majority of customers prefers another point to x, where β defines the behavior of indifferent customers. β-Condorcet points are called *Condorcet points* (Hansen & Thisse, 1981) for $\beta = 0$, *plurality points* (Wendell & McKelvey, 1981) for $\beta = 0.5$ and *majority points* (Wendell & Thorson, 1974) for $\beta = 1$.

It is well known that the existence of β-Condorcet points depends on the structure of the network and the distribution of the users (see, for instance, Bandelt, 1985; Hansen & Thisse, 1981; Hansen et al., 1986; Labbé, 1985). In tree networks, for example, the set of Condorcet points and medians is equal, so that there always exists at least one Condorcet point. In general networks, however, there can be a divergence between medians and Condorcet points and the latter need not exist. Bandelt (1985) characterizes the networks that always have a Condorcet point or a plurality point for any user distribution. He also answers the question as to when Condorcet points and medians coincide in those networks. A polynomial algorithm to determine the, possibly infinite, set of Condorcet points can, for example, be found in Hansen & Labbé (1988).

The Condorcet concept has been taken some steps further during the last decade. Campos Rodríguez & Moreno Pérez (2000a) define *α-Condorcet points* by allowing threshold values $\alpha \geq 0$ (and thus using an extended preference structure compared to the one used for the definition of Condorcet points) and setting $\beta = 0$ in (4). Obviously, there always exists a value for α, so that at least one α-Condorcet point exists. The authors give tight bounds for the relation of the objective function of median and center problems at an α-Condorcet point and the optimal values of the objective functions at a median or center in general networks and trees. These results extend similar well known results given in Hansen & Thisse (1981) and Labbé (1985) for Condorcet points. Campos Rodríguez & Moreno Pérez (2000b) modify the algorithm presented by Hansen & Labbé (1988) to obtain a polynomial algorithm to determine the set of α-Condorcet points for a given α.

The minimum α value that guarantees the existence of an α-Condorcet point is called tolerance distance. The corresponding α-Condorcet point, the *tolerant Condorcet point*, is introduced by Campos Rodríguez & Moreno Pérez (2000b). The authors provide a polynomial algorithm to compute the tolerance distance. One of their main theoretical results states, that every point at a distance less than α from a Condorcet point is an α-Condorcet point. However, this is not a necessary condition for α-Condorcet points.

Campos Rodríguez & Moreno Pérez (2003) extend the definition of α-Condorcet points to *$\alpha\gamma$-Condorcet points* by using

$$\pi([y \succ x]) + \beta\pi([x \sim y]) \leq \gamma\pi(N) \qquad \forall y \in N \tag{5}$$

instead of (4), where $0 \leq \gamma \leq 1$.[1] Thus, the proportion of customers needed to reject a location may be different from one half. If we fix only γ and compute the tolerance distance for this case we get *γ-tolerant Condorcet points*. Analogously, one may define *γ-Condorcet points* by setting $\alpha = 0$ or other point sets as, for example, *$\alpha\gamma$-plurality points* (setting

[1]Note that the term $\beta\pi([x \sim y])$ is not explicitly used by Campos Rodríguez & Moreno Pérez (2003), since $\beta = 0$ as it has been defined for α-Condorcet points. We include the term at this point to indicate that a β-value different from zero may still be considered, for instance by defining *$\alpha\beta\gamma$-Condorcet points*.

$\beta = 0.5$) (Campos Rodríguez & Moreno Pérez, 2008). Unfortunately, the algorithms and the majority of the results presented in Campos Rodríguez & Moreno Pérez (2003) and Campos Rodríguez & Moreno Pérez (2008) are restricted to the case, where the number of possible facility locations is a priori assumed to be finite, and thus lie beyond the scope of this review. Nevertheless, note that the point sets defined in this section are extended to include the case of multiple facilities in Campos Rodríguez & Moreno Pérez (2008). For example, a *p-Condorcet* set is a set of p locations such that there is no other set of p locations that is preferred by a strict majority of the customers.

Another contribution dealing with discrete versions of (single and multiple facility) $\alpha\gamma$-Condorcet and related problems is due to Noltemeier et al. (2007). The authors suggest an extension of (1)–(3) by defining more general *threshold functions* $\delta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$, where $\delta(0, y) = 0$ for all $y \in \mathbb{R}_0^+$ and δ is (weakly) monotonously increasing in both parameters. Given a threshold function δ and a parameter $\beta' \geq 0$, they replace (3) (and analogously (1) and (2)) by

$$[y \succ x] := \{u \in V | d(u, x) - \delta(\beta', d(u, x)) > d(u, y)\} \tag{6}$$

for the single facility case and give a straight forward extension of $\alpha\gamma$-Condorcet points to *$\beta'\gamma$-Condorcet points*.

So far, we have only been concerned with deterministic choice rules. Bauer et al. (1993) define a proportional choice rule and extend the concept of Condorcet points to what they call *k-optimal points*, where $k \in \mathbb{N} \cup \{\infty\}$ is a parameter of the choice rule. Let $x \neq y$ be any pair of points and $u_1, ..., u_n$ be the vertices of the network, then the probability for a customer located at u_i to purchase at x (rather than y) is given by

$$p^k_{u_i}(x, y) := \frac{d(u_i, y)^k}{d(u_i, x)^k + d(u_i, y)^k}. \tag{7}$$

For very large k the decisions of the customers are alike to the deterministic decision in the Condorcet case. The expected value of customers visiting facility x is

$$E^k(x, y) := \sum_{i=1}^{n} \pi(u_i) p^k_{u_i}(x, y) \tag{8}$$

and a k-optimal location x is defined to be a location where $E^k(x, y) \geq 0.5\pi(N)$ for all $y \in N$ with $y \neq x$. One of the main results of Bauer et al. (1993) states, that k-optimal points (if they exist) are always vertices of the network. Furthermore, the authors give a polynomial algorithm to determine all 1-optimal locations of a network.

Figure 3 summarizes some of the relations of the point sets that have been mentioned in this section so far.

We close this section by noting that Labbé (1990) defines *anti-Condorcet points* for the case where the players want to locate the facilities as far as possible from the users in the network. In this case no other point is allowed to be farther away from a strict majority of the users. The author presents a linear time algorithm for finding an anti-Condorcet point in trees. Furthermore, she gives a tight bound for the relation of the objective function of the antimedian problem at an anti-Condorcet point and at an antimedian. Finally, the

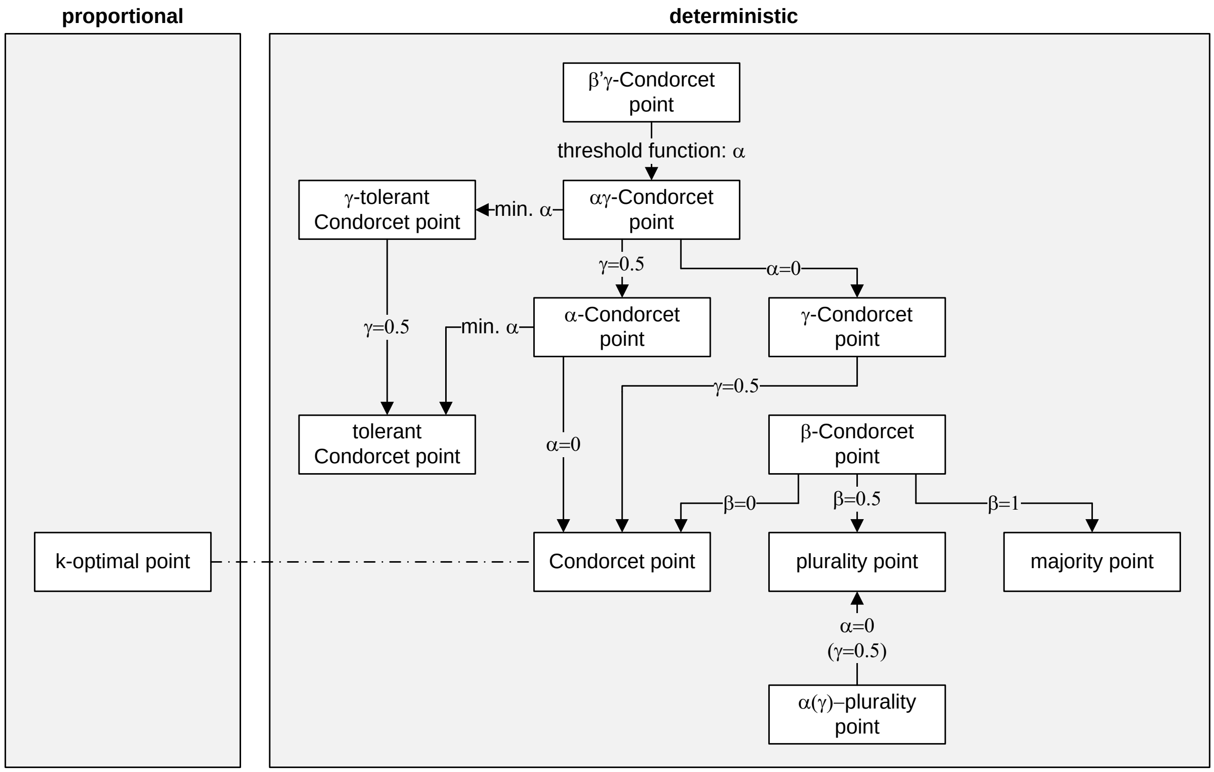

Figure 3: Condorcet points and related concepts.

α-anti-Condorcet point is defined analogously to the α-Condorcet point.

4. Voting location: Some (1|1)-centroid problems

Consider the preference structure (1)–(3) with $\alpha = 0$. As stated in the previous section, β-Condorcet points do not always exist in general networks. This leads to the incorporation of a minimax objective, i.e. the search for points such that the maximal number of users who prefer another point is minimal:

$$\min_{x \in N} \{\max_{y \in N} \{\pi([y \succ x]) + \beta\pi([x \sim y])\}\}. \tag{9}$$

An optimal solution to this optimization problem is called *β-Simpson point* (Bandelt, 1985). β-Simpson points are called *Simpson points* if $\beta = 0$ (Hansen & Labbé, 1988) and are originally due to Simpson (1969). If $\beta = 0.5$, then a β-Simpson point is called a *security point* (Slater, 1975). In the view of Hakimi's notion (Hakimi, 1983), the different kinds of Simpson points are equivalent to those (1|1)-centroids that consider the corresponding choice rules.

A general network contains at least one Simpson point. An algorithm of polynomial time complexity to determine the set of Simpson points of a general network is due to Hansen & Labbé (1988).

As in the case of α-Condorcet points, Campos Rodríguez & Moreno Pérez (2003) incorporate threshold values $\alpha \geq 0$ and set $\beta = 0$ to define *α-Simpson points*. Consider the definition of an $\alpha\gamma$-Condorcet point. An α-Simpson point is then defined to be an

$\alpha\gamma^*(\alpha)$-Condorcet point, where $\gamma^*(\alpha)$ is the smallest possible value that γ may take, such that the set of $\alpha\gamma$-Condorcet points is not empty. The definition of other point sets, as, for example, *α-security points* is straight forward (see Campos Rodríguez & Moreno Pérez, 2008). Again, note that Campos Rodríguez & Moreno Pérez (2008) and Noltemeier et al. (2007) consider the case of multiple facilities as well. Nevertheless, the results presented in Campos Rodríguez & Moreno Pérez (2003, 2008) and Noltemeier et al. (2007) are mainly concerned with discrete location problems and are thus omitted in this paper.

In their proportional model, Bauer et al. (1993) define the maximal relative *k-rejection* of a point $x \in N$ to be

$$\rho^k(x) := \sup_{x \neq y \in N} \frac{E^k(y,x)}{\pi(N)} \tag{10}$$

and, since k-optimal points need not exist in a general network, introduce *k-suboptimal points*. These are the locations of the network where the maximal relative k-rejection is minimal. The authors provide a polynomial algorithm to compute 1-suboptimal points of a network.

Figure 4 extends Figure 3 to give an overview of the point sets defined in this section so far.

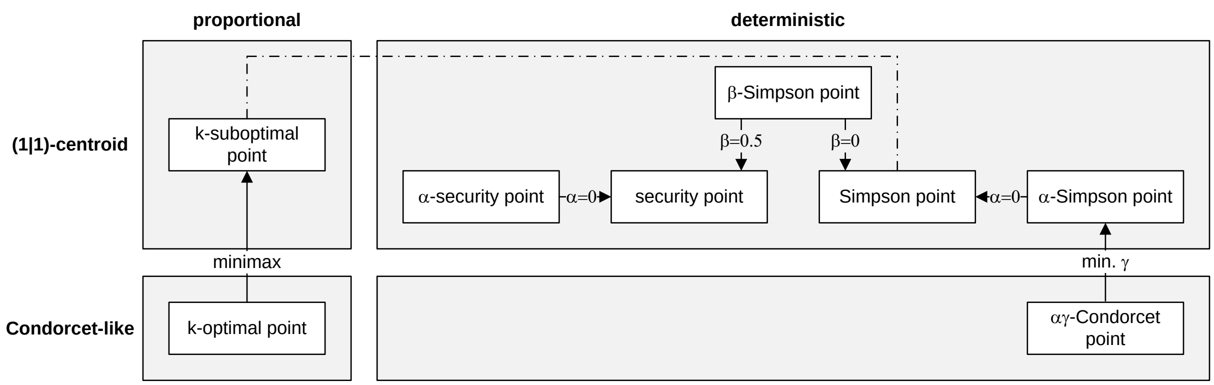

Figure 4: Some (1|1)-centroid problems related to the Condorcet concept.

Spoerhase & Wirth (2010) consider the preference structure (1)–(3) and introduce the concept of *monotonous gain functions* as a general model to describe several competitive and voting location problems. They consider tree networks $T = (V, E, \lambda)$ with $\pi : V \to \mathbb{Q}_0^+$, $\lambda : E \to \mathbb{Q}^+$ and $\lambda([u,v]) = d(u,x) + d(x,v)$ for any point x on an edge $[u,v] \in E$. A gain function $\Phi : T \times T \to \mathbb{Q}$ maps a pair of points (y,x) to the value $\Phi(y,x)$, denoted by $\Phi(y \succ x)$ as well, as a measure of the follower's influence at point y after the leader has located in point x. A gain function is called monotonous, if there is a function $\varphi : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ such that

1. $\Phi(y \succ x) = \varphi(\pi([y \succ x]), \pi([x \succ y]))$ for all points $(x,y) \in T$,
2. φ is monotonously increasing in the first parameter and monotonously decreasing in the second parameter,
3. φ can be evaluated in constant time.

Moreover, the authors define the *absolute Φ-score of a point x* to be $\Phi(x) := \max_{y \in T} \Phi(y \succ x)$ and the *absolute Φ-score of a tree* to be $\Phi^* := \min_{x \in T} \Phi(x)$. Any point $y \in T$ where

$\Phi(y \succ x) = \Phi(x)$ is called a Φ-*witness* of the leader point x. An *absolute* Φ-*solution* is a point $x \in T$ with $\Phi(x) = \Phi^*$. Therefore, the α-Simpson concept, for example, can be modeled by the monotonous gain function $\Gamma(y \succ x) := \pi([y \succ x])$. Thus, any absolute Γ-solution is an α-Simpson point. Spoerhase & Wirth (2010) derive an algorithm that finds an arbitrary element of the set of all absolute Φ-solutions for any monotonous gain function on a tree in time $O(n)$. The algorithm iteratively decreases the tree network, maintaining a so called leader tree which is always guaranteed to contain an absolute Φ-solution. Spoerhase & Wirth (2009a) extend these results (for $\pi : V \to \mathbb{R}_0^+$ and $\lambda : E \to \mathbb{R}^+$) by providing an algorithm of time complexity $O(n \log n)$ to compute the set of all absolute Φ-solutions for any monotonous gain function on a tree.

We close this section by noting that, in line with the anti-Condorcet concept, Labbé (1990) defines *anti-Simpson points* without giving any concrete results.

5. $(\mathbf{r}|\mathbf{X_p})$-medianoid problems

Consider the characteristics of a *basic* $(r|X_p)$-*medianoid problem* where the leader's facilities are located at distinct vertices: Customers are solely located at the vertices of the network and select the closest facility (binary choice) to accommodate their deterministic, inelastic demand. Well known results on the computational complexity of this problem are listed in Table 3 (see, for example, Eiselt & Laporte, 1996, for more details). Megiddo et al. (1983) show that an optimal solution to this basic $(r|X_p)$-medianoid problem can be determined in $O\left(n^r m^r / r!\right)$ time by discretizing the original network and applying an enumeration procedure.

Table 3: Some complexity results on a basic $(r|X_p)$-medianoid problem.

Problem characteristics	NP-hard	Polynomial
customers located at vertices binary choice (closest facility) deterministic, inelastic demand	$(r\|X_p)$ on general networks (Megiddo et al., 1983) $(r\|X_1)$ on general networks (Hakimi, 1983)	$(r\|X_p)$: $O(rn^2)$ on tree networks (Megiddo et al., 1983)

A similar discretization result for this basic $(r|X_p)$-medianoid problem has recently been presented by Suárez-Vega et al. (2004a) (cf. also Pelegrín et al., 2006, 2010; Santos-Peñate et al., 2007). The authors employ the concept of *isodistant points* in networks. This concept has been introduced by Peeters & Plastria (1998) (see below). A point x is said to be (u_j, f)-isodistant relative to a customer located at $u_j \in V$ with $\pi(u_j) \neq 0$ and an existing facility $f \in X_p$, if $d(u_j, x) = d(u_j, f)$. Any (u_j, f)-isodistant point for some $u_j \in V$ and $f \in X_p$ is called isodistant point. Pelegrín et al. (2010) present an algorithm of time complexity $O(n^3)$ to determine all isodistant points of a network for known values of $d(u_j, f)$ for all $u_j \in V$ and $f \in X_p$. For the basic $(r|X_p)$-medianoid problem under consideration, Suárez-Vega et al. (2004a) prove that the follower's market share is constant on each of the open segments of the network, that are defined by neighboring vertices or isodistant points. Therefore, one can discretize the $(r|X_p)$-medianoid problem by considering an arbitrary point on each of these segments as well as the isodistant points and vertices of the network.

Suárez-Vega et al. (2004a) consider extensions of the basic $(r|X_p)$-medianoid problem as well (cf. also Santos-Peñate et al., 2007). For example, they analyze the case of *elastic demand*. Let f_{u_j} be a nondecreasing and concave real function with $f_{u_j}(0) > 1$ for all $u_j \in V$. Then

$$MS_{\text{binary elastic}}(Y_r, X_p) := \sum_{u_j \in \{u_i \in V | D(u_i, Y_r) < D(u_i, X_p)\}} \frac{\pi(u_j)}{f_{u_j}(D(u_j, Y_r))} \tag{11}$$

defines the market share of the follower. The authors show that an optimal solution need not exist, so that we may seek ϵ-optimal solutions, i.e. solutions that guarantee an objective function value at most ϵ units away from a known upper bound. Furthermore, they show that an ϵ-optimal solution with all of the follower's facilities located at a vertex or sufficiently close to an isodistant point of the network always exists.

Well known results due to Hakimi (1986, 1990) are related to alternative modifications of the basic $(r|X_p)$-medianoid problem. The author presents a *partially binary choice rule with inelastic demand*, where f_{u_j} is a nondecreasing and concave real function with $f_{u_j}(0) > 0$ for all $u_j \in V$, i.e.

$$MS_{\text{part. binary inelastic}}(Y_r, X_p) := \sum_{u_j \in V} \frac{\pi(u_j) f_{u_j}(D(u_j, X_p))}{f_{u_j}(D(u_j, X_p)) + f_{u_j}(D(u_j, Y_r))}, \tag{12}$$

a *partially binary choice rule with elastic demand*, where f_{u_j} is a nondecreasing and concave real function with $f_{u_j}(0) > 1$ for all $u_j \in V$, i.e.

$$MS_{\text{part. binary elastic}}(Y_r, X_p) := \sum_{u_j \in V} \frac{\pi(u_j)(f_{u_j}(D(u_j, X_p)) - 1)}{\left[f_{u_j}(D(u_j, X_p)) f_{u_j}(D(u_j, Y_r)) - 1\right]}, \tag{13}$$

a *proportional choice rule with inelastic demand*, where f_{u_j} is a nondecreasing and concave real function with $f_{u_j}(0) > 0$ for all $u_j \in V$, i.e.

$$MS_{\text{prop. inelastic}}(Y_r, X_p) := \sum_{u_j \in V} \sum_{x_i \in Y_r} \frac{\frac{\pi(u_j)}{f_{u_j}(d(u_j, x_i))}}{\sum_{x_k \in Y_r \cup X_p} \frac{1}{f_{u_j}(d(u_j, x_k))}}, \tag{14}$$

and, finally, a *proportional choice rule with elastic demand*, where f_{u_j} is a nondecreasing linear function with $f_{u_j}(0) > 1$ for all $u_j \in V$, i.e.

$$MS_{\text{prop. elastic}}(Y_r, X_p) := \sum_{u_j \in V} \sum_{x_i \in Y_r} \frac{\frac{\pi(u_j)}{f_{u_j}(d(u_j, x_i)) - 1}}{1 + \sum_{x_k \in Y_p \cup X_p} \frac{1}{f_{u_j}(d(u_j, x_k)) - 1}}. \tag{15}$$

Hakimi (1986, 1990) proves the existence of an optimal solution Y_r^* with $Y_r^* \subset V$ to the $(r|X_p)$-medianoid problems in all of these cases (cf. also Bauer et al., 1993, for the proportional inelastic case). Note that in the latter two cases Hakimi assumes $p + r \leq n$. For the proportional elastic case, Suárez-Vega et al. (2004a) show that Hakimi's result remains true for f_{u_j} being nondecreasing concave real functions with $f_{u_j}(0) > 1$ for all $u_j \in V$. Moreover, they present computational results on the performance of some heuristic methods that are applied to problem instances of the discretized versions of the basic $(r|X_p)$-medianoid problem as well as it's extensions related to formulas (11)–(15).

Dasci et al. (2002) generalize the basic $(r|X_p)$-medianoid problem by additionally associating a value $\delta(e) = \delta([u, v])$ with edge $e = [u, v] \in E$. This value represents the uniform demand density of the edge, i.e. the unit demand per unit length. Thus, demand may exist at any point of the network. Dasci et al. (2002) motivate this assumption by noting that applications for problems with *edge demands* occur in a variety of fields. The demand for retail products in a city, for example, usually originates at the houses on a street rather than at the intersections of the streets. Since the basic problem with vertex demand only is NP-hard (see Table 3), the generalized version with additional edge demand is obviously NP-hard too. The authors show that the problem is still NP-hard if vertex demands are dropped, by reducing the Dominating Set problem (cf. Garey & Johnson, 1979) to the resulting $(r|X_p)$-medianoid problem with edge demand only. Furthermore, they show that a simple augmentation of the discretization result of Megiddo et al. (1983) can be applied to the $(r|X_p)$-medianoid problem with edge (or vertex and edge) demands. This augmentation needs to take into account that the problem might not have an optimal solution, so that we may seek ϵ-optimal solutions. As a result, Dasci et al. (2002) are able to derive an algorithm of polynomial time complexity to compute ϵ-optimal solutions of the resulting $(1|X_p)$-medianoid problem.

Another contribution dealing with a $(1|X_p)$-medianoid problem with uniform demand densities $\delta(e)$ on edges $e \in E$ is Okunuki & Okabe (2002). The authors assume that N is planar and that $O(p) < O(n) = O(m)$. Furthermore, consumers with deterministic, inelastic demand probabilistically choose among facilities. The probabilistic choice rule is based on the (network) *Huff-model*. According to this model, the probability of a customer visiting a facility is proportional to the attractiveness (for example the size) of the facility and inversely proportional to some non decreasing, positive function g_{dist} of the distance to it.[2] Define a_i to be a measure of the attractiveness of facility $i = 1, ..., p+1$ and let $\lambda > 0$. The proposed model of consumer choice then defines the probability for a customer located at point x to visit facility $i \in 1, ..., p+1$ as follows:

$$p(x, x_i) := \frac{a_i e^{-\lambda d(x,x_i)}}{\sum\limits_{j=1}^{p+1} a_j e^{-\lambda d(x,x_j)}}. \tag{16}$$

The expected value of customers visiting facility i located at x_i is

$$E(x_i) := \sum_{[u,v]\in E} \int\limits_{x=0}^{\lambda([u,v])} p(x, x_i)\delta([u, v])dx. \tag{17}$$

[2]For details on the Huff-model and related customer choice models, see the references in Okunuki & Okabe (2002) or the reviews on consumers in location models by Drezner & Eiselt (2002) and spatial interaction modeling by Roy & Thill (2004).

Okunuki & Okabe (2002) derive an algorithm of time complexity $O(n^2 \log n)$ to find approximately optimal solutions to the corresponding $(1|X_p)$-medianoid problem. The algorithm is based on transformations of the original network and numerical methods.

The Huff-model (with a concave function g_{dist}) is also applied to a $(1|X_p)$-medianoid problem with vertex demand only by Peeters & Plastria (1998), who point to an anomaly of this proportional choice rule: Customers will patronize very distant facilities even when there exist more attractive facilities that are much closer. The authors therefore propose the usage of a modified Huff-model, the *Pareto-Huff-model*. In this model, the customers located at vertex $u_j \in V$ will only visit a subset of facilities, denoted by $Par_{u_j}(X_p \cup \{x_{p+1}\})$, that are Pareto optimal with respect to the attractiveness and the distance:

$$f \in Par_{u_j}(X_p \cup \{x_{p+1}\}) \Leftrightarrow \forall g \in X_p \cup \{x_{p+1}\} \backslash \{f\} \begin{cases} \text{either } a_g < a_f, \\ \text{or } d(u_j, g) > d(u_j, f), \\ \text{or } a_g = a_f \\ \text{and } d(u_j, g) = d(u_j, f). \end{cases} \tag{18}$$

Demand is assumed to be deterministic and inelastic. While there always exists an optimal solution at a vertex for the regular Huff-model, the same does not hold for the Pareto-Huff model. Peeters & Plastria (1998) derive a discretization result which states that there is always an optimal solution at, or an ϵ-optimal solution close to, a vertex or an isodistant point of the network for any $\epsilon > 0$. The authors show that there exist at most $2m\hat{n}p$ isodistant points in a network, where $\hat{n}$ is the number of vertices $v \in V$ with $\pi(v) \neq 0$, and propose an $O(\hat{n}n \log n + pm\hat{n}^2)$ enumeration algorithm to determine all (ϵ-)optimal solutions of the problem. Pelegrín et al. (2010) extend these results by proving that the discretization result holds for the case of multiple facilities, i.e. $r \geq 1$, as well.

Suárez-Vega et al. (2007b) present another model that incorporates attractiveness levels of facilities in a network with vertex demand only. Let $f_{u_j} : \mathbb{R}_0^+ \to \mathbb{R}$ be an increasing and concave function for all $u_j \in V$ and $A_p = (a_1, ..., a_p)$ and $A_r = (a_{p+1}, ..., a_{p+r})$ be the attractiveness levels of the facilities. Furthermore, associate an *attraction threshold* τ_{u_j} with every vertex $u_j \in V$, such that the users at u_j patronize facility i at point $x_i \in N$ with attractiveness level a_i only if $a_i / f_{u_j}(d(u_j, x_i)) \geq \tau_{u_j}$. We denote a subset $S_L(u_j)$ of a set L of facility locations that capture demand at node $u_j \in V$ by $S_L(u_j) := \{x_k \in L | \frac{a_k}{f_{u_j}(d(u_j,x_k))} \geq \tau_{u_j}\}$. Then

$$MS_{\text{attr.}}(Y_r, A_r, X_p, A_p) := \sum_{u_j \in \{u_i \in V | S_{Y_r}(u_i) \neq \emptyset\}} \pi(u_j) \frac{\sum\limits_{x_i \in S_{Y_r}(u_j)} \frac{a_i}{f_{u_j}(d(u_j,x_i))}}{\sum\limits_{x_i \in S_{Y_r \cup X_p}(u_j)} \frac{a_i}{f_{u_j}(d(u_j,x_i))}} \tag{19}$$

defines the follower's market share. Given an attractiveness level a_i and a vertex $u_j \in V$, a point $x \in N$ is said to be a (u_j, τ_{u_j}, a_i)-*threshold point*, if the condition $f_{u_j}(d(u_j, x)) = a_i / \tau_{u_j}$ holds. Each element of the set

$$T(a_i) = \cup_{u_j \in V} \{x \in N | x \text{ is a } (u_j, \tau_{u_j}, a_i)\text{-threshold point}\} \tag{20}$$

is called a threshold point with respect to the attractiveness level a_i. Suárez-Vega et al. (2007b) prove that, given X_p, A_p and A_r, there exists an optimal solution Y^* to the $(r|X_p)$-medianoid problem under consideration, where every facility $x_i^* \in Y^*$ with attractiveness level a_i is a vertex or an element of $T(a_i)$.

While so far the models have considered location decisions based on given attractiveness levels, Suárez-Vega et al. (2004b) define both, locations and attractiveness levels of the follower, as decision variables (cf. also Santos-Peñate et al., 2007). Thus, they define what they call a *$(r|X_p, A_p)$-medianoid problem*: Given the attractiveness levels A_p and locations X_p of the leader, the follower rationally chooses attractiveness levels A_r and locations X_r. Obviously, one needs to include a cost of attractiveness in this case. The authors do so by including fixed cost of attractiveness $F(a_j)$, where F is a positive, continuous and non-decreasing function of a non-negative real variable, and maximizing profit, i.e. market share minus the sum of attractiveness cost for all facilities. They analyze inelastic vertex demand under binary, partially binary and proportional choice rules, i.e. extensions of the basic $(r|X_p)$-medianoid problem and the problems related to (12) and (14). Suárez-Vega et al. (2004b) provide discretization results for the partially binary and proportional choice rules, while in the case of binary customer choice a discretization can only be applied if attractiveness levels are given. Furthermore, they present some heuristic algorithms and computational results for the resulting discrete problems. Suárez-Vega et al. (2007a) extend these results by providing a discretization result for the $(r|X_p, A_p)$-medianoid problem related to (15) (proportional, elastic customer choice), where f_{u_j} are nondecreasing and concave real functions for all $u_j \in V$.

García Pérez et al. (2000) (cf. also García Pérez & Pelegrín Pelegrín, 1997) examine a $(1|X_1)$-medianoid problem on a tree network with vertex customers only. The follower's potential locations are restricted to a given path within the tree network. Demand is assumed to be inelastic and customers select the closest facility (binary choice). The entrant is assumed to have two main objectives, the first is to maximize market share, the second is to minimize the maximum travel time of the customers that are being served by the entrant. This *bicriteria optimization model* might have applications in situations where the time to attend a demand is important, as, for example, in messenger delivery services or fast food services. The authors assume travel time to be proportional to distance and show that the problem on a tree network can be reduced to the same type of problem on a chain network. They define a location to be efficient, if there exists no other location with equally large or larger market share and equally large or smaller maximum travel time. However, since the infimum value of the maximum travel time might not be reached for some of the provably finite number of values of possible market shares of the follower, García Pérez et al. (2000) consider ϵ-efficient locations, that are defined in line with ϵ-optimality, as well. Finally, the authors derive an algorithm to determine the entire set of efficient and ϵ-efficient locations.

García et al. (2010) incorporate prices into a $(r|X_p)$-medianoid problem on a network.[3] As, in most markets, the choice of location is less flexible than the choice of prices,

[3]Note that García et al. (2010) assume that multiple players are initially located in the network. These players may be interpreted as a single opponent from the point of view of the entrant. Thus, we may interpret the problem setting as a $(r|X_p)$-medianoid problem.

locational models usually assume that simultaneous *price competition* occurs after the location decisions have been made. Thus, García et al. (2010) allow the players to change prices in a simultaneous pricing game (Betrand game) after the follower's entry. The follower anticipates the price competition in this separate stage when making the location decisions. The players are assumed to be profit maximizers and employ spatial price discrimination. Customers are located at the vertices of the network and patronize the facility that offers the lowest price (binary choice). The price sensitive demand at a vertex $u_j \in V$ of the network is described by a continuous and strictly decreasing function $q_{u_j}(p)$ of the price $p \in [0, p_{u_j}^{max}]$, where $p_{u_j}^{max}$ is the reservation price of the users at u_j. Note that spatial price discrimination is applied to $(r|X_p)$-medianoid problems with constant vertex demand by Pelegrín et al. (2006) (for $r = 1$) and Pelegrín et al. (2010) as well. García et al. (2010) assume that the players are able to price below $p_{u_j}^{max}$ for each $u_j \in V$. Furthermore, they assume that the marginal delivered cost (the sum of marginal cost of production and transportation) at each $u_j \in V$ is independent of the amounts sold and that the players will not price below their marginal delivered cost. Therefore, a separate Betrand game takes place for every vertex of the network for any fixed set of locations. García et al. (2010) show that each player will monopolize the set of vertices where his marginal delivered cost is lower than the minimum marginal delivered cost of the other players. The players then either set the optimal monopoly price or the price equal to the minimum marginal delivered cost of the competitors in each of their monopolized vertices and a price equal to their minimum marginal delivered cost in the other vertices. Thus, the location price problem is reduced to a location problem if all players are assumed to set these equilibrium prices. The authors then additionally assume the marginal production cost to be a positive concave function on an edge of the network and the marginal transportation cost to be a positive, concave and increasing function with respect to the distance of a facility to a vertex of the network. They prove that, under these assumptions, there exists a set of vertices which is an optimal solution to the $(r|X_p)$-medianoid problem.

We close this section by extending Table 3 in listing complexity results that have been explicitly mentioned in the surveyed papers in Table 4. Note that discretization results might be exploited to derive further efficient algorithms (especially for $r = 1$).

Table 4: Some complexity results on selected $(r|X_p)$-medianoid problems.

Problem characteristics	NP-hard	Polynomial
(vertex and) edge demand binary choice (closest facility) deterministic, inelastic demand	$(r\|X_p)$ on general networks (Dasci et al., 2002)	$(1\|X_p)$: $O(nm^2)$ ϵ-opt. solution on general networks (Dasci et al., 2002)
edge demand proportional choice (Huff model) deterministic, inelastic demand		$(1\|X_p)$: $O(n^2 \log n)$ approx. opt. solution on general networks (Okunuki & Okabe, 2002)
vertex demand Pareto-Huff model deterministic, inelastic demand		$(1\|X_p)$: $O(\hat{n}n \log n + pm\hat{n}^2)$ ϵ-opt. solution on general networks (Peeters & Plastria, 1998)

6. (r|p)-centroid problems

Hakimi (1983) was the first to publish complexity results on the *basic* $(r|p)$*-centroid* *problem*, which we define in line with the basic $(r|p)$-medianoid problem, i.e. we consider vertex customers who patronize the closest facility to accommodate their deterministic, inelastic demand. He proved the basic $(1|p)$-centroid problem on a general network to be NP-hard by reduction of the Vertex Cover problem (cf. Garey & Johnson, 1979). Another well known result is due to Hansen & Labbé (1988) who provided an algorithm of polynomial time complexity to determine the set of all optimal solutions to the basic $(1|1)$-centroid problem on general networks (cf. already Section 4).

Table 5: Some complexity results on a basic $(r|p)$-centroid problem.

Problem characteristics	NP-hard	Polynomial
customers located at vertices binary choice (closest facility) deterministic demand inelastic demand	$(1\|p)$ on general networks (Hakimi, 1983) $(1\|p)$ on pw-bounded networks (Spoerhase & Wirth, 2009b) $(r\|p)$ on chain networks (Spoerhase & Wirth, 2009b)	$(1\|1)$: $O(n^4 m^2 \log(mn) \log(\pi(N)))$ on general networks (Hansen & Labbé, 1988) $(1\|p)$: $O(n^2 (\log n)^2 \log \pi(N) \log \hat{D})$ on tree networks (Spoerhase & Wirth, 2009b)

These results have recently been extended by Spoerhase & Wirth (2009b). The authors provide a NP-hardness proof for the basic $(r|p)$-centroid problem on chain networks that uses a reduction of the Partition problem (cf. Garey & Johnson, 1979). Furthermore, they investigate the basic $(1|p)$-centroid problem. They show that Hakimi's result on its complexity on general networks remains true on networks of bounded pathwidth (pw-bounded networks), while a discretization result can be utilized to design an exact algorithm of polynomial time complexity on tree networks. Note that, differing from our network notation, the authors assume $\pi : V \to \mathbb{Q}_0^+$, $\lambda : E \to \mathbb{Q}^+$ and $\lambda([u, v]) = d(u, x) + d(x, v)$ for any point x on an edge $[u, v] \in E$. The complexity results are summarized in Table 5.

Based on Table 5 and Section 5, one may conclude that centroid problems are substantially harder than medianoid problems (cf. also Section 8). This is supported by Hakimi (1983), who defines an *approximate solution* to a $(r|p)$-centroid problem as a solution with an objective function value that is within a fixed positive (integer) factor α of being optimal, where α is not a function of the size of the network. He proves that even the problem of finding an approximate solution to the basic $(1|p)$-centroid problem is NP-hard. Spoerhase & Wirth (2008) extend this result by showing that there exists a fully polynomial time approximation scheme for the basic $(r|p)$-centroid problem on chain networks with $\pi : V \to \mathbb{N}$ and $\lambda : E \to \mathbb{N}^+$.

Motivated by the fact that the basic $(r|p)$-centroid problem is polynomially time solvable on tree networks for $r = p = 1$, research in this field has focused mainly on incorporating additional features into this special case. Shiode & Drezner (2003), for example, extend the basic $(1|1)$-centroid problem on a tree network by assuming that the vertex weights at the follower's time of entry used by the leader in his decision are *stochastic*; that is, they are random variables according to mutually independent normal distributions $\pi(u_j) \sim N(\mu_j, \sigma_j^2)$ with mean μ_j and variance σ_j^2 for all $u_j \in V$. The follower then locates his facility knowing the location of the leader and the vertex demands. Note that the authors do not permit co-location, i.e. locating at the same point of the network, of the two players. The leader wishes to minimize the maximum total demand taken away by the follower with a satisfying probability β with $0.5 < \beta < 1$. It is easy to see that there always exists an optimal location of the follower at a vertex of the tree network.

Shiode & Drezner (2003) additionally prove that there always exists an optimal solution to this stochastic (1|1)-centroid problem at a vertex of the tree network and present a simple efficient solution procedure based on an enumeration of the vertices and bisection.

An alternative extension to the basic (1|1)-centroid problem on tree networks is presented by García Pérez & Pelegrín Pelegrín (2003). The authors incorporate parametric prices, i.e. they assume that the unit mill price of the product is fixed for each player. Transportation cost is assumed to be a linear function of distance. The demand at each node is served by the lower cost facility (parametric price + transportation cost). Moreover, the authors assume that each player gets a positive profit once located at an optimal location, so that competition is possible. Hereafter, they present two $O(n^3 \log n)$ algorithms to generate the set X_1^* of all optimal leader locations under the assumption that the parametric prices are different. Note that the algorithm presented by Spoerhase & Wirth (2009a) (cf. Section 4) may be applied to compute the set X_1^* in $O(n \log n)$ time. García Pérez & Pelegrín Pelegrín (2003) determine this set for equal parametric prices as well. Furthermore, they prove that there always exists an optimal location for the follower at a vertex of the tree network when prices are different, while this does not have to be the case for the leader. When prices are equal, there always exists an optimal location for the leader at a vertex, while this might not be true for the follower.

García Pérez & Pelegrín Pelegrín (1997) analyze the basic (1|1)-centroid problem on a tree network for different objective functions. They, first, consider the maximization of market share and, second, the minimization of the maximum travel time of the customers. They prove that, if we restrict the players' locations to a path within the tree network, the game can be reduced to a corresponding game on a chain network and that the optimal locations of both, the leader and the follower, can be determined analytically.

While García Pérez & Pelegrín Pelegrín (1997) study the variation of objective functions that affect both players, Berman & Gavious (2007) analyze a two player leader follower game on a network with differing objectives of the players. The leader, a state, locates p facilities that provide support in case of a terrorist attack. Locating an emergency facility incurs a fixed cost $C > 0$. Moreover, the state decides on investing resources c_{prev} on prevention, such that the state strategy is a vector $s = (x_1, ..., x_p, c_{prev})$. The follower, a terrorist, attacks one of the cities, modeled as vertices of the network. For the case of more than one leader facility the authors analyze discrete versions of the problem, while results for a continuous version are presented for the case of one facility ((1|1)-centroid problem). The vertex weights of the graph are interpreted as the expected damage in case of a terrorist attack on that city. Edge weights represent the delay of shipment of resources from the facility. The terrorist may use a mixed strategy $t = (p_1, ..., p_n)$ where p_i is the probability that the terrorist will attack vertex $u_i \in V$. It is assumed that $\sum_{i=1}^{n} p_i = 1$ and $p_i \geq 0$ for all i. The probability for an attack to succeed is modeled by a continuous decreasing convex function $P(c_{prev})$ with $P(0) > 0$ and $\lim_{c_{prev} \to \infty} P(c_{prev}) = 0$. The disutility in case of a successful terrorist attack on city u_i is

$$f_i := (\alpha d(x, i) + \eta_i)\pi(u_i) \tag{21}$$

where x is the closest facility to the city and $\alpha > 0$ represents the cost of delaying one unit of resource due to a unit of distance. Furthermore, $\eta_i := \eta + \alpha d(i)$ with $\eta > 0$ and

$d(i)$ representing the delay at the city itself. Thus, the expected utility of the leader is described by

$$U_S(t, s) := -P(c_{prev}) \sum_{i=1}^{n} p_i f_i - pC - c_{prev}. \tag{22}$$

The terrorist's utility is defined to be

$$U_T(t, s) := \sum_{i=1}^{n} p_i(\gamma d(x, i) + \delta_i) w_i \tag{23}$$

with $\delta_i := \delta + \alpha d(i)$ and $\delta > 0$, $\gamma > 0$. Berman & Gavious (2007) show that an optimal solution to the $(1|1)$-centroid problem is included in a finite set of points, called local centers, that can be determined analytically. Moreover, they show that an optimal solution can be determined efficiently.

7. Multiple players, endogenous location and selected $\mathbb{R}^1$ models

So far, we have been able to interpret most of the games using Hakimi's framework of $(r|X_p)$-medianoid and $(r|p)$-centroid problems with problem specific assumptions and extensions. This is no longer reasonable if we consider more complex games with, for example, more than two players or endogenized market entry or location order. These kinds of assumptions, as suggested by Eiselt & Laporte (1996) for future research, have mainly been applied to linear markets. Table 6 lists some selected recent contributions.

Granot et al. (2010) allow an endogenously determined, potentially indefinite number of players to locate one facility on a line and, in an extension to this model, a network that may contain loops. Players locate in the location space, at fixed cost, according to an exogenously given order as long as it is profitable for them to do so. The demand of the uniformly spread customers is assumed to be elastic and modeled by a demand intensity function, denoted by $d(x)$, which is continuous and decreasing in the distance x to the closest facility (binary choice). Thus, in the linear model (and analogously in the network model) the revenue of a player, located at x_0, from the costumers in a subinterval $(x_0, x_0 + r)$, provided he is the closest player to these customers, is

$$D(r) := \int_0^r d(x)dx. \tag{24}$$

Prices are fixed to 1. Making some assumptions with regard to the demand intensity function and not permitting co-location, the authors show in their main result for the network model that, in equilibrium, players will first locate at the vertices of the network according to some easily computable order, and subsequently, they will locate on the edges. This result can be generalized to the case of edge specific demand intensity functions. Moreover, Granot et al. (2010) prove that the optimal number of facilities to be located by a monopolist is strictly smaller than the number of players who will be located in equilibrium of the competitive model if edge lengths are not too small.

Other features that have mainly been considered in sequential competitive location models on a line include asymmetric information (concerning, for example, customer preferences or the cost structure of the opponent), asymmetric (direction dependent) transportation cost, special settings in the pricing game (concerning the timing of events) and players with different objectives (global or local players, public or private firms, etc.). We list some related publications in Table 7.[4] The reader may find interesting ideas to be adopted in general network models in future.

8. On the influence of specific modeling assumptions

As pointed out by Eiselt & Laporte (1996), sequential competitive location problems are very sensitive with respect to the underlying modeling assumptions, especially in terms of complexity, the existence and nature of solutions and discretization results. D'Aspremont et al. (1979), for example, show that there is no Nash equilibrium in pure strategies in Hotelling's (first location then price) game with linear transportation cost.[5] However, by assuming the transportation cost to increase quadratically with the distance, the authors are able to derive a proof for the existence of a Nash equilibrium in which the competitors locate at the opposite ends of the line segment. Other examples are related to voting theory: In Section 3 we have illustrated that the existence of Condorcet points (and related concepts) often depends on the structure of the underlying network and the distribution of the users. This led to the incorporation of minimax objectives (Section 4) or a threshold value α. Sections 5–7, however, have mainly been describing the modeling assumptions themselves in detail. It is the aim of this section to explore the above-mentioned sensitivity of these models. In doing so, we assume the reader to be familiar with the effects summarized in Eiselt & Laporte (1996). Moreover, we note that the papers that have been subject of Sections 5 and 6 are mainly concerned with existence, complexity and discretization issues, while research on $\mathbb{R}^1$ models (Section 7) concentrates on effects like agglomeration, maximal differentiation, leader and follower advantages and the existence of equilibria. Therefore, this section is divided into two parts, highlighting the implications of general network models and models on a line separately.

Consider an arbitrary $(r|p)$-centroid problem on a network. Due to the sequential nature of the game, we are (in mathematical programming terms) faced with a bilevel (or hierarchical) optimization problem (see Bard, 1998, for an introduction to bilevel programming). The fact that even the linear bilevel programming problem in continuous variables is NP-hard in the strong sense (Bard, 1998, chap. 5) suggests, that we are unlikely to be confronted with polynomially time solvable problems. Indeed, as Table 5 demonstrates, the basic $(r|p)$-centroid problem is NP-hard even if we consider relatively simple network structures as, for example, chain networks. By fixing $r = 1$, one can derive an efficient algorithm on tree networks; A feature which is lost when considering a slightly more complex network class: pathwidth bounded networks. A polynomial time algorithm for the basic $(r|p)$-centroid problem on general networks is only known for the case $r = p = 1$. Thus, two of the major factors influencing the complexity of centroid

[4]Note that some of the authors analyze extensions to their models while we refer to their main models in the Table 7.

[5]Note that Hotelling (1929) claimed that, in equilibrium, the two competitors will agglomerate in the center of the line segment.

Table 6: Selected models on $\mathbb{R}^1$ with multiple players or endogenous location order.

		players		pricing and cost			customers		
authors	space	number	loc. order	pricing	setup/prod. cost	transp. cost	distribution	demand	choice
Aiura (2010)	$[-1, 1]$	3	exogenous	none	none	linear	uniform on unknown interval of length 1, multi period model with updated beliefs	inelastic	binary
Götz (2005)	$[0, 1]$	endogenous	exogenous	mill price set simult.	fixed setup cost	quadratic	uniform with variable density	inelastic	binary
Harter (1997)	$[0, 2]$	n (exogenous)	exogenous	mill price set simult.	none	quadratic	uniform on unknown interval of length 1	inelastic	binary
Heywood & Ye (2009a,b)	$[0, 1]$	n (exogenous), different ”types“	exogenous	spatial price discrimination	none	linear	uniform	inelastic	binary
Lambertini (1997)	$\mathbb{R}^1$	2	endogenous (simult. entry possible)	mill price, timing endogenous	none	quadratic	uniform on interval of length 1	inelastic	binary
Loertscher & Muehlheusser (2008)	two $[0, 1]$ markets	3, different ”types“	exogenous	none	none	linear	uniform with market specific density	inelastic	binary
Meza & Tombak (2009)	$[0, l]$, $l \in \mathbb{R}^+$	2	endogenous (simult. entry possible)	mill price set simult.	marginal prod. cost asymmetries	quadratic	uniform	inelastic	binary
Nilssen (1997)	$[0, 1]$	3	exogenous	none	none	linear, direction dependent	uniform	inelastic	binary
Rothschild et al. (2007)	$[0, 1]$	3, strategic contracts, merging	exogenous	spatial price discrimination	none	linear	uniform	inelastic	binary
Yates (1997)	$[0, 1]$	n (exogenous)	exogenous	none	none	linear, direction dependent	uniform	elastic	binary

Refer to the caption underneath Table 7 for an illustration of the cell entries.

Table 7: Other selected $\mathbb{R}^1$ models (continues on next page).

		players		pricing and cost			customers		
authors	space	number	loc. order	pricing	setup/prod. cost	transp. cost	distribution	demand	choice
Beladi et al. (2010)	$[0,1]$	2, vertically related	exogenous	spatial price discrimination	none	linear	uniform	inelastic or elastic (multiple products)	binary
Bonein & Turolla (2009)	$\mathbb{R}^1$	2	exogenous	mill price set simult.	none	quadratic	uniform on interval of length 1, asymmetric information	inelastic	binary
Boyer et al. (2003a) (cf. also (Boyer et al., 1995))	$[0,l]$, $l \in \mathbb{R}^+$	2	exogenous	mill price set simult.	marginal. prod. cost asymmetries, asymmetric information	quadratic	uniform	inelastic	binary
Boyer et al. (2003b) (cf. also (Boyer et al., 1995))	$[0,l]$, $l \in \mathbb{R}^+$	2	exogenous	mill price set simult. or spatial price discrimination	marginal. prod. cost asymmetries, asymmetric information, fixed setup cost	linear, quadratic	density fctn.	inelastic	binary
Dasci & Laporte (2005, 2007)	$[0,1]$	2, endogenous number of facilities	exogenous	none	fixed setup cost	linear	density fctn.	inelastic	binary
Fleckinger & Lafay (2010)	$[0,1]$	2	exogenous	mill price set at the same time as loc. decision	none	general fctn.	density fctn.	inelastic	binary
Lai (2001)	$[0,1]$	2	exogenous	none	none	linear, direction dependent	uniform	elastic	binary
Lambertini (2002)	$[0,1]$	2, uncertainty over follower's time of entry	exogenous	mill price set simult.	none	quadratic	uniform	inelastic	binary
Matsumura & Matsushima (2003)	$[0,1]$	2, different "types"	exogenous	mill price set simult., price regulation	none	quadratic	uniform	inelastic	binary
Matsumura & Matsushima (2010)	$[0,1]$	2	exogenous	mill price set simult.	marginal prod. cost can be changed by investment in a pregame stage	quadratic	uniform	inelastic	binary
Nilssen & Sørgard (2002)	$[0,1]$	2, different "types"	exogenous	none	none	linear, direction dependent	uniform	elastic	binary

Table 7: (continued)

		players		pricing and cost			customers		
authors	space	number	loc. order	pricing	setup/prod. cost	transp. cost	distribution	demand	choice
Peng & Tabuchi (2007a,b)	$[0, 1]$	2, endogenous number of facilities, endogenous number of product varieties	exogenous	parametric price	fixed cost per variety	quadratic	uniform	elastic	binary
Rhee (2006)	$[0, 1]$	2	exogenous	mill price set simult.	none	quadratic	uniform, customers differ along unobservable characteristics	inelastic	binary
Tabuchi & Thisse (1995)	$\mathbb{R}^1$	2	exogenous	mill price set simult.	none	quadratic	uniform or triangular on $[0, 1]$	inelastic	binary
Tyagi (1999)	$\mathbb{R}^1$	2	exogenous	mill price, endogenous order	none	quadratic	uniform on bounded interval	inelastic	binary
Tyagi (2000)	$\mathbb{R}^1$	2	exogenous	mill price set simult.	marginal prod. cost asymmetries	quadratic	uniform on $[-0.5, 0.5]$	inelastic	binary
Zhou & Vertinsky (2001)	$[0, 1]$	2	exogenous	mill price set simult., monopoly price before follower's entry, interest rate	fixed setup cost (may be different for players)	quadratic	uniform, time dependent density	inelastic	binary

Columns:	space	specification of location space
	number	information on number of players; endogenously or exogenously determined; "types", e.g. public or private, domestic or foreign firms; contracts among firms; vertically related industries; product varieties (refer to the papers for details)
	loc. order	location order endogenously or exogenously determined
	pricing	type of pricing subgame
	setup/prod. cost	information on setup or production cost
	transp. cost	information on transportation cost; linear or quadratic function of distance; dependent on or independent of direction of travel
	distribution	type of user distribution
	demand	elastic or inelastic demand
	choice	choice rule

problems are the network structure and the number of facilities to be located by the players.[6] Tables 3 and 4 show that this result carries over to the basic $(r|X_p)$-medianoid problem if we treat r as an arbitrary input parameter. On tree networks, we are able to efficiently determine an optimal solution for arbitrary values of r, while this is not the case on general networks. Furthermore, the influence of the number of facilities on the complexity persists when considering additional features as, for example, edge demand (Dasci et al., 2002) or other types of choice rules (Okunuki & Okabe, 2002).

It is well known that modeling assumptions influence the existence and cardinality of finite dominating sets of competitive location problems. This, in turn, may effect the applicability of enumerative algorithms or the neighborhood definitions of potential heuristic algorithms. In the case of $(r|X_p)$-medianoid problems (see Section 5), such factors include:

- Binary choice: While there exists a subset of the vertex set (frequently referred to as the "Hakimi property" or "node optimality") that corresponds to an optimal solution in the case of choice rules (12)–(15) (partially binary and proportional choice with inelastic and elastic demand, respectively), finite dominating sets are usually substantially larger in the case of binary choice rules. This is true for both, inelastic and elastic demand (basic $(r|X_p)$-medianoid problem and choice rule (11)). Furthermore, it remains true when considering $(r|X_p, A_p)$-medianoid problems under different choice rules (Suárez-Vega et al., 2007a).

- Pareto optimality: The same effect arises when considering (ϵ-optimal solutions of) the Pareto-Huff-model instead of the Huff-model (Peeters & Plastria, 1998).

- Edge demand: When (additionally) incorporating edge demand, one may need to seek ϵ-optimal solutions (Dasci et al., 2002). The resulting effect on the cardinality of finite dominating sets is rather small.

Discretization results concerning $(r|p)$-centroid problems (Section 6) are rather limited. A basic $(r|1)$-centroid of a general network is always a vertex for $r \geq 2$, while this is not the case for $r = 1$. On tree networks, however, there always exists a vertex that is a $(1|1)$-centroid (see Hakimi, 1990, and the references therein). The latter statement remains true in the case of stochastic demand as presented by Shiode & Drezner (2003), but it may be wrong when parametric prices are included (García Pérez & Pelegrín Pelegrín, 2003). An additional discretization result for the case $r = p = 1$ is presented by Berman & Gavious (2007). Furthermore, Spoerhase & Wirth (2009b) derive a (non polynomial) discretization result for the basic $(1|p)$-centroid problem on a tree network.

As mentioned above, modeling assumptions do not only influence the complexity and discretization results of the underlying problem, but may also effect the existence and nature of its solutions. In Section 5, for example, due to the potential non-existence of optimal solutions, ϵ-optimal solutions had to be considered in the case of choice rule (11), when considering edge demand or when applying the Pareto-Huff model. Results

[6]The dependency of complexity on network structure is well known in the (competitive and non-competitive) location literature. Kincaid (2011) gives an overview of classical contributions in this field.

concerning $\mathbb{R}^1$ models are listed in Table 9. Those models are typically designed with regard to analytic traceability or analyzed by simulation. Complexity issues are usually not considered in detail.

Table 9: $\mathbb{R}^1$ - Main features and results (continues on next page).

Authors	Main features	Results
	Demand uncertainty	
Aiura (2010)	Demand uncertainty in a multi period model: Customers distributed on an unknown interval of length 1. Later entrants may adjust predictions on the distribution by observing demand signals arising after preceding firms enter the market.	Agglomeration of the three firms occurs (only) if firms can observe signals on the customer distribution. In this case they dislike revealing the signals and therefore chose the same location as the preceding firm.
Bonein & Turolla (2009)	Customers distributed on an interval of length 1. One player is perfectly informed while the other faces demand uncertainty.	Demand uncertainty leads to differentiation when faced by the leader while it is an agglomeration force when faced by the follower. Welfare losses are higher in the latter case.
Harter (1997)	Customers distributed on an unknown interval of length 1.	Except from the duopoly case, firms locate almost evenly throughout the location space.
	Customer attributes	
Götz (2005)	Uniform customer distribution with variable density (market size).	Locational patterns are in general asymmetric in the duopoly case. Profits are non-monotonic in market size. Equilibrium profits of all firms may be larger in situations in which more firms are active.
Rhee (2006)	Consumers differ both in their locations and in their tastes along unobservable characteristics (random component in customer's utility functions).	The more the customer choices depend on the unobservable characteristics, the closer to the center of the market the follower locates. Anticipating this behavior, the leader locates farther away from the center. Not only the locations, but also the degree of first mover advantage (or disadvantage) is determined by the level of dependence of the customer behavior on unobservable characteristics.
Zhou & Vertinsky (2001)	Time dependent customer density. Inclusion of interest rate. Leader monopoly prior to follower's entry.	The follower always maximally differentiates. The leader chooses the center of the market if transportation cost and market growth rate are sufficiently low and interest rate and fixed cost are sufficiently large. Otherwise the leader will locate as far away from the center as possible.
	Different "types" of players	
Beladi et al. (2010)	Vertically related industry with one upstream and two downstream firms who move sequentially. The latter produce two out of three differentiated goods each.	A merged firm (one of the downstream firms and the upstream firm) will locate farther away from the social optimum than the firm outside the merger.
Heywood & Ye (2009a,b)	Sequential games with a domestic public firm, domestic private firms and foreign firms. Domestic welfare is defined to be consumer surplus plus domestic profit and equals global welfare minus the profit of the foreign firm.	Without considering the entry of a foreign firm, the presence of a public firm generally increases welfare. The entry of a foreign firm often lowers domestic welfare. Privatization of the public firm may lower domestic welfare but can increase global welfare.
Loertscher & Muehlheusser (2008)	A global player competes with local players on two (heterogenous) linear markets with the same product.	Agglomeration occurs across markets.
Matsumura & Matsushima (2003)	Sequential games with a public and a private firm. The effect of price regulation is considered.	The desirable role (leader or follower) of the public firm depends on the presence of price regulation. Neither price regulation nor privatization of the public firm improves welfare.
Rothschild et al. (2007)	Possibility of strategic contracts or merging between two out of three firms that enter sequentially.	The outcomes in terms of welfare consequences and location choice are substantially different according to different modes of entry.
	Endogenous timing	
Lambertini (1997)	The timing of entry and pricing is endogenously determined. Players may locate outside the consumer interval.	The (subgame perfect) equilibrium in pure strategies is reached by playing sequentially in the price stage after having set locations simultaneously.

Table 9: (continued)

Authors	Main features	Results
Meza & Tombak (2009)	Asymmetric firms with different marginal production cost. The timing of entry is endogenously determined.	The existence of an equilibrium in pure strategies, the timing of events and the locations themselves depend on the marginal cost differences. Usually, sequential market entry naturally arises, the low cost firm takes the role of the leader and the follower maximally differentiates.
	Asymmetric transportation cost	
Lai (2001)	Transportation cost is finite only in one direction of travel.	No (subgame perfect) equilibrium in pure strategies.
Nilssen (1997)	Asymmetric transportation cost.	Locational patterns (as well as a potential first mover advantage) depend on the explicit values of the transportation costs. There exists no indifference among late movers that want to locate between early movers.
Nilssen & Sørgard (2002)	Asymmetric transportation cost. Public leader, private follower.	Both, triggered differentiation and triggered agglomeration, may be welfare maximizing (depending on the asymmetry of transportation costs).
Yates (1997)	Transportation cost is finite only in one direction of travel.	Equilibria are symmetric in the sense that all players obtain the same market share. Note that, in contrast to Lai (2001), Yates (1997) considers ϵ-optimal locations.
	Information asymmetry	
Boyer et al. (2003a,b)	Information asymmetry concerning marginal production cost.	Information asymmetry may not cause any distortion in the location game if a fixed cost of entry is not present. But, in general, information plays a crucial role in location strategies preventing the entry of a follower.
Lambertini (2002)	An infinite time horizon is included. The follower's time of entry is uncertain. As long as the follower has not entered, the leader acts as a monopolist.	The later the follower is expected to enter, the closer to the center of the market the leader locates.
Tyagi (2000)	Marginal production cost asymmetries. Players may locate outside the consumer interval.	Unless the leader is certain that the follower will not have a superior cost structure, it may be better to locate away from the market center.
	Pregame investments	
Matsumura & Matsushima (2010)	There is an ex ante marginal production cost difference (efficient leader, inefficient follower). Marginal production cost can be changed by investment in a pregame stage.	An ex ante inefficient follower has a stronger incentive to invest, given that the ex ante cost difference is sufficiently large. A decrease in cost of this follower pushes the facilities apart, resulting in increasing prices and market share of the follower. Decreasing cost of the ex ante efficient leader lead to decreasing differentiation and decreasing prices.
	Special pricing game	
Fleckinger & Lafay (2010)	The players (simultaneously or sequentially) choose price and location at the same time. A general transportation cost function is applied.	There is no equilibrium in the simultaneous game while there exist equilibria in the sequential case. The follower charges a higher price and always earns more than the leader. Differentiation is never minimal (agglomeration) nor maximal.
Tyagi (1999)	The order of setting prices after the location stage is endogenized. Players may locate outside the consumer interval.	The unique equilibrium outcome involves players choosing sequential pricing over simultaneous pricing. The leader in location acts as the leader in prices.
	Location outside the consumer interval	
Tabuchi & Thisse (1995)	Triangular consumer density. Players may locate outside the consumer interval.	For a symmetric triangular density, only asymmetric equilibria (in pure strategies) exist (even for simultaneous choice of locations). A first mover advantage is induced by allowing location outside the consumer interval.
	Location densities	

Table 9: (continued)

Authors	Main features	Results
Dasci & Laporte (2005, 2007)	The players open an endogenously determined (fixed cost) number of facilities. Strategies are defined in terms of their location densities (no precise representation of the locations).	The model is solvable analytically. If the follower enters the market, he opens at least as many facilities as the leader. The leader can effectively deter entry (even in the case of a cost disadvantage). However, this may not always be rational. Entry and entry deterrence decisions are quite sensitive to consumer densities and fixed cost. This is not the case for the location strategies, once both players are present.
	Product varieties	
Peng & Tabuchi (2007a,b)	Competition in location and product varieties. Endogenization of the number of facilities and varieties of the players by including a fixed cost per variety.	If each player is restricted to locating a single facility, neither agglomeration nor maximum differentiation occur. Allowing multiple stores yields a rich set of equilibrium outcomes.

9. Conclusion

In this paper we have presented a review of recent developments in the field of sequential competitive location problems with a focus on problems defined on networks. We have included the class of voting location problems. In the latter context, we have given an overview of different kinds of point sets in networks, as, for example, generalizations of the well known Condorcet and Simpson sets, and their interrelationships. We have outlined the features and main results of recently published $(r|X_p)$-medianoid and $(r|p)$-centroid problems. Moreover, we have presented a tabular overview of more complex games that have mainly been defined on linear markets. These models feature, for example, an endogenized market entry or location order, multiple players, asymmetric information, asymmetric transportation cost or different player objectives. Finally, we have outlined the effects of some modeling assumptions on the complexity, the existence and nature of solutions and discretization results of the problems under consideration. We refer to Younies & Eiselt (2011) for ideas on future research directions.

References

Aikens, C. H. (1985). Facility location models for distribution planning. *European Journal of Operational Research*, *22*, 263–279.

Aiura, H. (2010). "Signal-jamming" leads to "minimum differentiation" under demand uncertainty. *The Annals of Regional Science*, *45*, 245–271.

Anderson, S. P., de Palma, A., & Thisse, J.-F. (1992a). *Discrete Choice Theory of Product Differentiation*. Cambridge: MIT Press.

Anderson, S. P., de Palma, A., & Thisse, J.-F. (1992b). Social surplus and profitability under different spatial pricing policies. *Southern Economic Journal*, *58*, 934–949.

Avella, P., Benati, S., Cánovas Martinez, L., Dalby, K., Di Girolamo, D., Dimitrijevic, B., Ghiani, G., Giannikos, I., Guttmann, N., Hultberg, T. H., Fliege, J., Marin, A., Munõz Márquez, M., Ndiaye, M. M., Nickel, S., Peeters, P., Pérez Brito, D., Policastro, S., Saldanha de Gama, F. A., & Zidda, P. (1998). Some personal views on the current state and the future of locational analysis. *European Journal of Operational Research*, *104*, 269–287.

Bandelt, H.-J. (1985). Networks with Condorcet solutions. *European Journal of Operational Research*, *20*, 314–326.

Bard, J. F. (1998). *Practical Bilevel Optimization - Algorithms and Applications*. Dordrecht: Kluwer.

Bauer, A., Domschke, W., & Pesch, E. (1993). Competitive location on a network. *European Journal of Operational Research*, *66*, 372–391.

Beladi, H., Chakrabarti, A., & Marjit, S. (2010). Sequential spatial competition in vertically related industries with different product varieties. *Economics Letters*, *106*, 112–114.

Berman, O., Drezner, Z., & Krass, D. (2010). Generalized coverage: New developments in covering location models. *Computers & Operations Research*, *37*, 1675–1687.

Berman, O., & Gavious, A. (2007). Location of terror response facilities: A game between state and terrorist. *European Journal of Operational Research*, *177*, 1113–1133.

Bonein, A., & Turolla, S. (2009). Sequential location under one-sided demand uncertainty. *Research in Economics*, *63*, 145–159.

Boyer, M., Laffont, J.-J., Mahenc, P., & Moreaux, M. (1995). Sequential location equilibria under incomplete information. *Economic Theory*, *6*, 323–350.

Boyer, M., Mahenc, P., & Moreaux, M. (2003a). Asymmetric information and product differentiation. *Regional Science and Urban Economics*, *33*, 93–113.

Boyer, M., Mahenc, P., & Moreaux, M. (2003b). Entry preventing locations under incomplete information. *International Journal of Industrial Organization*, *21*, 809–829.

Brandeau, M. L., & Chiu, S. S. (1989). An overview of representative problems in location research. *Management Science*, *35*, 645–674.

Campos Rodríguez, C. M., & Moreno Pérez, J. A. (2000a). Comparison of α-Condorcet points with median and center locations. *TOP*, *8*, 215–234.

Campos Rodríguez, C. M., & Moreno Pérez, J. A. (2000b). The tolerant Condorcet points. *Studies in Locational Analysis*, *14*, 173–190.

Campos Rodríguez, C. M., & Moreno Pérez, J. A. (2003). Relaxation of the Condorcet and Simpson conditions in voting location. *European Journal of Operational Research*, *145*, 673–683.

Campos Rodríguez, C. M., & Moreno Pérez, J. A. (2008). Multiple voting location problems. *European Journal of Operational Research*, *191*, 437–453.

Current, J., Daskin, M., & Schilling, D. (2002). Discrete network location models. In Z. Drezner, & H. W. Hamacher (Eds.), *Facility Location - Applications and Theory* (pp. 81–118). Berlin: Springer.

Dasci, A. (2011). Conditional location problems on networks and in the plane. In H. A. Eiselt, & V. Marianov (Eds.), *Foundations of Location Analysis* (pp. 179–206). New York: Springer.

Dasci, A., Eiselt, H. A., & Laporte, G. (2002). On the (r, X_p)-medianoid problem on a network with vertex and edge demands. *Annals of Operations Research*, *111*, 271–278.

Dasci, A., & Laporte, G. (2005). A continuous model for multistore competitive location. *Operations Research*, *53*, 263–280.

Dasci, A., & Laporte, G. (2007). Erratum to "A continuous model for multistore competitive location". *Operations Research*, *55*, 1188–1190.

Daskin, M. S. (1995). *Network and Discrete Location - Models, Algorithms, and Applications*. New York: Wiley.

Daskin, M. S. (2008). What you should know about location modeling. *Naval Research Logistics*, *55*, 283–294.

D'Aspremont, C., Gabszewicz, J. J., & Thisse, J.-F. (1979). On Hotelling's "Stability in competition". *Econometrica*, *47*, 1145–1150.

Domschke, W., & Drexl, A. (1985). *Location and Layout Planning - An International Bibliography*. Berlin: Springer.

Domschke, W., & Krispin, G. (1997). Location and layout planning. *OR Spectrum*, *19*, 181–194.

Drezner, T. (1995a). Competitive facility location in the plane. In Z. Drezner (Ed.), *Facility Location - A Survey of Applications and Methods* (pp. 285–300). New York: Springer.

Drezner, Z. (Ed.) (1995b). *Facility Location - A Survey of Applications and Methods*. New York: Springer.

Drezner, Z., & Eiselt, H. A. (2002). Consumers in competitive location models. In Z. Drezner, & H. W. Hamacher (Eds.), *Facility Location - Applications and Theory* (pp. 150–178). Berlin: Springer.

Drezner, Z., & Hamacher, H. (Eds.) (2002). *Facility Location - Applications and Theory*. Berlin: Springer.

Eiselt, H. A. (1993). Competition in locational models. *Studies in Locational Analysis*, *5*, 129–147.

Eiselt, H. A., & Laporte, G. (1989). Competitive spatial models. *European Journal of Operational Research*, *39*, 231–242.

Eiselt, H. A., & Laporte, G. (1996). Sequential location problems. *European Journal of Operational Research*, *96*, 217–231.

Eiselt, H. A., Laporte, G., & Thisse, J.-F. (1993). Competitive location models: A framework and bibliography. *Transportation Science*, *27*, 44–54.

Eiselt, H. A., & Marianov, V. (Eds.) (2011). *Foundations of Location Analysis*. New York: Springer.

Erkut, E., & Neuman, S. (1989). Analytical models for locating undesirable facilities. *European Journal of Operational Research*, *40*, 275–291.

Fleckinger, P., & Lafay, T. (2010). Product flexibility and price competition in Hotelling's duopoly. *Mathematical Social Sciences*, *60*, 61–68.

Francis, R. L., & Goldstein, J. M. (1974). Location theory: A selective bibliography. *Operations Research*, *22*, 400–410.

Friesz, T. L., Miller, T., & Tobin, R. L. (1988). Competitive network facility location models: A survey. *Papers of the Regional Science Association*, *65*, 47–57.

García, M. D., Pelegrín, B., & Fernández, P. (2010). Location strategy for a firm under competitive delivered prices. *The Annals of Regional Science*, *doi: 10.1007/s00168-010-0372-x*.

García Pérez, M. D., Fernández Hernández, P., & Pelegrín Pelegrín, B. (2004). On price competition in location-price models with spatially separated markets. *TOP*, *12*, 351–374.

García Pérez, M. D., Lafuente Lechuga, M., & Pelegrín Pelegrín, B. (2000). Efficient locations to optimize market share and maximum travel time in a duopoly competition model. *Studies in Locational Analysis*, *14*, 83–100.

García Pérez, M. D., & Pelegrín Pelegrín, B. (1997). Two-stage competitive location on a tree. *Studies in Locational Analysis*, *11*, 101–114.

García Pérez, M. D., & Pelegrín Pelegrín, B. (2003). All Stackelberg location equilibria in the Hotelling's duopoly model on a tree with parametric prices. *Annals of Operations Research*, *122*, 177–192.

Garey, M. R., & Johnson, D. S. (1979). *Computers and Intractability - A Guide to the Theory of NP-Completeness*. New York: Freeman.

Götz, G. (2005). Endogenous sequential entry in a spatial model revisited. *International Journal of Industrial Organization*, *23*, 249–261.

Graitson, D. (1982). Spatial competition à la Hotelling: A selective survey. *The Journal of Industrial Economics*, *31*, 13–25.

Granot, D., Granot, F., & Raviv, T. (2010). On competitive sequential location in a network with a decreasing demand intensity. *European Journal of Operational Research*, *205*, 301–312.

Hakimi, S. L. (1983). On locating new facilities in a competitive environment. *European Journal of Operational Research*, *12*, 29–35.

Hakimi, S. L. (1986). p-median theorems for competitive locations. *Annals of Operations Research*, *6*, 77–98.

Hakimi, S. L. (1990). Locations with spatial interactions: Competitive locations and games. In P. B. Mirchandani, & R. L. Francis (Eds.), *Discrete Location Theory* (pp. 439–478). New York: Wiley.

Hale, T., & Moberg, C. R. (2003). Location science research: A review. *Annals of Operations Research*, *123*, 21–35.

Hamacher, H. W., & Nickel, S. (1998). Classification of location models. *Location Science*, *6*, 229–242.

Handler, G. Y., & Mirchandani, P. B. (1979). *Location on Networks - Theory and Algorithms*. Cambridge: MIT Press.

Hansen, P., & Labbé, M. (1988). Algorithms for voting and competitive location on a network. *Transportation Science*, *22*, 278–288.

Hansen, P., & Thisse, J.-F. (1981). Outcomes of voting and planning: Condorcet, Weber and Rawls locations. *Journal of Public Economics*, *16*, 1–15.

Hansen, P., Thisse, J.-F., & Wendell, R. E. (1986). Equivalence of solutions to network location problems. *Mathematics of Operations Research*, *11*, 672–678.

Hansen, P., Thisse, J.-F., & Wendell, R. E. (1990). Equilibrium analysis for voting and competitive location problems. In P. B. Mirchandani, & R. L. Francis (Eds.), *Discrete Location Theory* (pp. 479–501). New York: Wiley.

Harter, J. F. R. (1997). Hotelling's competition with demand location uncertainty. *International Journal of Industrial Organization*, *15*, 327–334.

Hay, D. A. (1976). Sequential entry and entry-deterring strategies in spatial competition. *Oxford Economic Papers*, *28*, 240–257.

Heywood, J. S., & Ye, G. (2009a). Mixed oligopoly and spatial price discrimination with foreign firms. *Regional Science and Urban Economics*, *39*, 592–601.

Heywood, J. S., & Ye, G. (2009b). Mixed oligopoly, sequential entry, and spatial price discrimination. *Economic Inquiry*, *47*, 589–597.

Hooker, J. N., Garfinkel, R. S., & Chen, C. K. (1991). Finite dominating sets for network location problems. *Operations Research*, *39*, 100–118.

Hotelling, H. (1929). Stability in competition. *The Economic Journal*, *39*, 41–57.

Kincaid, R. K. (2011). Exploiting structure: Location problems on trees and treelike graphs. In H. A. Eiselt, & V. Marianov (Eds.), *Foundations of Location Analysis* (pp. 315–334). New York: Springer.

Klose, A., & Drexl, A. (2005). Facility location models for distribution system design. *European Journal of Operational Research*, *162*, 4–29.

Labbé, M. (1985). Outcomes of voting and planning in single facility location problems. *European Journal of Operational Research*, *20*, 299–313.

Labbé, M. (1990). Location of an obnoxious facility on a network: A voting approach. *Networks*, *20*, 197–207.

Lai, F.-C. (2001). Sequential locations in directional markets. *Regional Science and Urban Economics*, *31*, 535–546.

Lambertini, L. (1997). Unicity of the equilibrium in the unconstrained Hotelling model. *Regional Science and Urban Economics*, *27*, 785–798.

Lambertini, L. (2002). Equilibrium locations in a spatial model with sequential entry in real time. *Regional Science and Urban Economics*, *32*, 47–58.

Loertscher, S., & Muehlheusser, G. (2008). Global and local players in a model of spatial competition. *Economics Letters*, *98*, 100–106.

Love, R. F., Morris, J. G., & Wesolowsky, G. O. (1988). *Facilities Location - Models & Methods*. New York: North-Holland.

Matsumura, T., & Matsushima, N. (2003). Mixed duopoly with product differentiation: Sequential choice of location. *Australian Economic Papers*, *42*, 18–34.

Matsumura, T., & Matsushima, N. (2010). When small firms fight back against large firms in R&D activities. *The B.E. Journal of Economic Analysis and Policy*, *10*, Article 81.

McGinnis, L. F. (1977). A survey of recent results for a class of facilities location problems. *IIE Transactions*, *9*, 11–18.

Megiddo, N., Zemel, E., & Hakimi, S. L. (1983). The maximum coverage location problem. *SIAM Journal on Algebraic and Discrete Methods*, *4*, 253–261.

Melo, M. T., Nickel, S., & Saldanha-da-Gama, F. (2009). Facility location and supply chain management - a review. *European Journal of Operational Research*, *196*, 401–412.

Mesa, J. A., & Boffey, T. B. (1996). A review of extensive facility location in networks. *European Journal of Operational Research*, *95*, 592–603.

Meza, S., & Tombak, M. (2009). Endogenous location leadership. *International Journal of Industrial Organization*, *27*, 687–707.

Miller, T. C., Friesz, T. L., & Tobin, R. L. (1996). *Equilibrium Facility Location on Networks*. Berlin: Springer.

Mirchandani, P. B., & Francis, R. L. (Eds.) (1990). *Discrete Location Theory*. New York: Wiley.

Nagy, G., & Salhi, S. (2007). Location-routing: Issues, models and methods. *European Journal of Operational Research*, *177*, 649–672.

Nilssen, T. (1997). Sequential location when transportation costs are asymmetric. *Economics Letters*, *54*, 191–201.

Nilssen, T., & Sørgard, L. (2002). A public firm challenged by entry: duplication or diversity? *Regional Science and Urban Economics*, *32*, 259–274.

Noltemeier, H., Spoerhase, J., & Wirth, H.-C. (2007). Multiple voting location and single voting location on trees. *European Journal of Operational Research*, *181*, 654–667.

Okunuki, K.-I., & Okabe, A. (2002). Solving the Huff-based competitive location model on a network with link-based demand. *Annals of Operations Research*, *111*, 239–252.

Osborne, M. J. (1995). Spatial models of political competition under plurality rule: A survey of some explanations of the number of candidates and the positions they take. *Canadian Journal of Economics*, *28*, 261–301.

Owen, S. H., & Daskin, M. S. (1998). Strategic facility location: A review. *European Journal of Operational Research*, *111*, 423–447.

Peeters, P. H., & Plastria, F. (1998). Discretization results for the Huff and Pareto-Huff competitive location models on networks. *TOP*, *6*, 247–260.

Pelegrín, B., Fernández, P., Suárez, R., & García, M. D. (2006). Single facility location on a network under mill and delivered pricing. *IMA Journal of Management Mathematics*, *17*, 373–385.

Pelegrín, B., Suárez-Vega, R., & Cano, S. (2010). Isodistant points in competitive network facility location. *TOP*, *doi: 10.1007/s11750-010-0148-6*.

Peng, S.-K., & Tabuchi, T. (2007a). Erratum: Spatial competition in variety and number of stores. *Journal of Economics and Management Strategy*, *16*, 537.

Peng, S.-K., & Tabuchi, T. (2007b). Spatial competition in variety and number of stores. *Journal of Economics and Management Strategy*, *16*, 227–250.

Plastria, F. (1995). Continuous location problems. In Z. Drezner (Ed.), *Facility Location - A Survey of Applications and Methods* (pp. 225–262). New York: Springer.

Plastria, F. (2001). Static competitive facility location: An overview of optimisation approaches. *European Journal of Operational Research*, *129*, 461–470.

Plastria, F. (2002). Continuous covering location problems. In Z. Drezner, & H. W. Hamacher (Eds.), *Facility Location - Applications and Theory* (pp. 37–79). Berlin: Springer.

Prescott, E. C., & Visscher, M. (1977). Sequential location among firms with foresight. *The Bell Journal of Economics*, *8*, 378–393.

ReVelle, C. S., & Eiselt, H. A. (2005). Location analysis: A synthesis and survey. *European Journal of Operational Research*, *165*, 1–19.

ReVelle, C. S., Eiselt, H. A., & Daskin, M. S. (2008). A bibliography for some fundamental problem categories in discrete location science. *European Journal of Operational Research*, *184*, 817–848.

Rhee, B.-D. (2006). First-mover disadvantages with idiosyncratic consumer tastes along unobservable characteristics. *Regional Science and Urban Economics*, *36*, 99–117.

Rothschild, R., Heywood, J. S., & Monaco, K. (2007). Strategic contracts versus multiple plants: Location under sequential entry. *The Manchester School*, *75*, 237–257.

Roy, J. R., & Thill, J.-C. (2004). Spatial interaction modelling. *Papers in Regional Science*, *83*, 339–361.

Şahin, G., & Süral, H. (2007). A review of hierarchical facility location models. *Computers & Operations Research*, *34*, 2310–2331.

Santos-Peñate, D. R., Suárez-Vega, R., & Dorta-González, P. (2007). The leader–follower location model. *Networks and Spatial Economics*, *7*, 45–61.

Serra, D., & ReVelle, C. (1995). Competitive location in discrete space. In Z. Drezner (Ed.), *Facility Location - A Survey of Applications and Methods* (pp. 367–386). New York: Springer.

Shiode, S., & Drezner, Z. (2003). A competitive facility location problem on a tree network with stochastic weights. *European Journal of Operational Research*, *149*, 47–52.

Simpson, P. B. (1969). On defining areas of voter choice: Professor Tullock on stable voting. *The Quarterly Journal of Economics*, *83*, 478–490.

Slater, P. J. (1975). Maximin facility location. *Journal of Research of the National Bureau of Standards, Sect. B*, *79*, 107–115.

Smith, H. K., Laporte, G., & Harper, P. R. (2009). Locational analysis: Highlights of growth to maturity. *Journal of the Operational Research Society*, *60, Supplement 1*, 140–148.

Snyder, L. V. (2006). Facility location under uncertainty: A review. *IIE Transactions*, *38*, 547–564.

Spoerhase, J., & Wirth, H.-C. (2008). Approximating (r,p)-centroid on a path. In *Proceedings of the 7th Cologne-Twente Workshop on Graphs and Combinatorial Optimization, CTW'08* (pp. 193–195). Gargnano, Italy.

Spoerhase, J., & Wirth, H.-C. (2009a). Optimally computing all solutions of Stackelberg with parametric prices and of general monotonous gain functions on a tree. *Journal of Discrete Algorithms*, *7*, 256–266.

Spoerhase, J., & Wirth, H.-C. (2009b). (r,p)-centroid problems on paths and trees. *Theoretical Computer Science*, *410*, 5128–5137.

Spoerhase, J., & Wirth, H.-C. (2010). Relaxed voting and competitive location under monotonous gain functions on trees. *Discrete Applied Mathematics*, *158*, 361–373.

von Stackelberg, H. (1934). *Marktform und Gleichgewicht*. Vienna: Springer.

Suárez-Vega, R., Dorta-González, P., & Santos-Peñate, D. R. (2007a). The $(r|X_p)$-medianoid model for non-essential services and proportional preferences with attraction functions. *Studies in Locational Analysis*, *16*, 1–13.

Suárez-Vega, R., Santos-Peñate, D. R., & Dorta-González, P. (2004a). Competitive multifacility location on networks: The $(r|X_p)$-medianoid problem. *Journal of Regional Science*, *44*, 569–588.

Suárez-Vega, R., Santos-Peñate, D. R., & Dorta-González, P. (2004b). Discretization and resolution of the $(r|X_p)$-medianoid problem involving quality criteria. *TOP*, *12*, 111–133.

Suárez-Vega, R., Santos-Peñate, D. R., & Dorta-González, P. (2007b). The follower location problem with attraction thresholds. *Papers in Regional Science*, *86*, 123–137.

Suárez-Vega, R., Santos-Peñate, D. R., Dorta-González, P., & Rodríguez-Díaz, M. (2010). A multi-criteria GIS based procedure to solve a network competitive location problem. *Applied Geography*, *doi: 10.1016/j.apgeog.2010.06.002*.

Swamy, M. N. S., & Thulasiraman, K. (1981). *Graphs, Networks, and Algorithms*. New York: Wiley.

Tabuchi, T., & Thisse, J.-F. (1995). Asymmetric equilibria in spatial competition. *International Journal of Industrial Organization*, *13*, 213–227.

Tansel, B. C., Francis, R. L., & Lowe, T. J. (1983a). Location on networks: A survey. Part I: The p-center and p-median problems. *Management Science*, *29*, 482–497.

Tansel, B. C., Francis, R. L., & Lowe, T. J. (1983b). Location on networks: A survey. Part II: Exploiting tree network structure. *Management Science*, *29*, 498–511.

Tyagi, R. K. (1999). Pricing patterns as outcomes of product positions. *Journal of Business*, *72*, 135–157.

Tyagi, R. K. (2000). Sequential product positioning under differential costs. *Management Science*, *46*, 928–940.

Wendell, R. E., & McKelvey, R. D. (1981). New perspectives in competitive location theory. *European Journal of Operational Research*, *6*, 174–182.

Wendell, R. E., & Thorson, S. J. (1974). Some generalizations of social decisions under majority rule. *Econometrica*, *42*, 893–912.

Yates, A. J. (1997). Hotelling and the New York stock exchange. *Economics Letters*, *56*, 107–110.

Younies, H., & Eiselt, H. A. (2011). Sequential location models. In H. A. Eiselt, & V. Marianov (Eds.), *Foundations of Location Analysis* (pp. 163–178). New York: Springer.

Zhou, D., & Vertinsky, I. (2001). Strategic location decisions in a growing market. *Regional Science and Urban Economics*, *31*, 523–533.