# A GENERALIZED METHOD FOR THE DERIVATION OF NON-LINEAR STATE-SPACE MODELS FROM CIRCUIT SCHEMATICS

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**ABSTRACT**

Digital emulation of analog circuits for musical audio processing, like synthesizers, guitar effect pedals, or vintage amplifiers, is an ongoing research topic. David Yeh proposed to use the nodal DK method to derive a non-linear state-space system from a circuit schematic in a very systematic way. However, this approach has some drawbacks and limitations, especially with respect to the modeling of individual circuit elements. Therefore, in this paper, we present an alternative that is more flexible than the nodal DK method and hopefully allows for easier integration of almost arbitrary element models. This flexibility and generality in our opinion outweighs the relatively small cost associated with it in terms of increased matrix sizes. We therefore believe the proposed method to be a useful tool for circuit simulation.

***Index Terms*—** circuit analysis, circuit simulation, virtual analog modeling, state-space model

## 1. INTRODUCTION

Digital emulation of analog circuits for musical audio processing, like synthesizers, guitar effect pedals, or vintage amplifiers, is an ongoing research topic. While the direct design of a wanted audio effect in the digital domain surely is a reasonable way to go, often the replication of an analog circuit's sound is desired, to e.g. obtain the exact same sound known from an admired musician. One way to achieve this is to analyze the analog circuit, derive a mathematical model for it, and then transform this model into executable code. Besides ad-hoc approaches, the most commonly applied models are wave digital filters and state-space models, where this paper solely deals with the latter.

In [1–3], Yeh proposes to use the nodal DK method to derive a non-linear state-space system from a circuit schematic in a very systematic way. The method has been successfully applied with minor extensions in numerous works, e.g. [4–7]. The derived non-linear state-space systems have the form

$$\boldsymbol{x}(n) = \boldsymbol{A}\boldsymbol{x}(n-1) + \boldsymbol{B}\boldsymbol{u}(n) + \boldsymbol{C}\boldsymbol{i}(n) \tag{1a}$$

$$\boldsymbol{y}(n) = \boldsymbol{D}\boldsymbol{x}(n-1) + \boldsymbol{E}\boldsymbol{u}(n) + \boldsymbol{F}\boldsymbol{i}(n) \tag{1b}$$

$$\boldsymbol{v}(n) = \boldsymbol{G}\boldsymbol{x}(n-1) + \boldsymbol{H}\boldsymbol{u}(n) + \boldsymbol{K}\boldsymbol{i}(n) \tag{1c}$$

$$\boldsymbol{i}(n) = \boldsymbol{f}\big(\boldsymbol{v}(n)\big), \tag{1d}$$

where $\boldsymbol{u}(n)$ is the input vector of the system, $\boldsymbol{y}(n)$ is the output vector, $\boldsymbol{x}(n)$ is the vector of the system states, and $\boldsymbol{i}(n)$ and $\boldsymbol{v}(n)$ are vectors of the current through and voltages across non-linear elements. The coefficient matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{F}$, $\boldsymbol{G}$, $\boldsymbol{H}$, and $\boldsymbol{K}$ can be derived systematically from the circuit by applying the nodal-DK method, and the non-linear function $\boldsymbol{f}$ is simply the collection of the voltage-current relationships of all non-linear elements in the circuit. For every time-step $n$, first a suitable $\boldsymbol{i}(n)$ needs to be found such that the result of (1c) is consistent with (1d), which may then be used in (1a) and (1b).

The nodal-DK method first applies time-discretization to all stateful elements (e.g. capacitors and inductors), typically by employing the trapezoidal integration rule (to be explained in section 4), to derive so-called companion circuits. These companion circuits contain a current-source which drives a current depending on the previous time-step's state. The resulting time-discretized circuit is then subject to the (modified) nodal analysis by treating all non-linear elements as independent current sources to obtain (1a)–(1c). Finally, the dependence of the non-linear element currents on the voltages is reinstated by adding (1d).

Unfortunately, treating the non-linear elements as independent current sources during the derivation and applying nodal analysis has some drawbacks and limitations. First, all nodes of the circuit need to be connected in some way. This, for example, may not be the case if a transformer connects two otherwise separate parts of the circuit. Second, any node where only non-linear elements are connected to will be mapped into a node with only independent current sources connected. This must in general be considered inconsistent and hence, does not possess a solution. Third, not all elements may be easily represented as (controlled) current sources. For example, a clipping op amp is naturally modelled as a non-linear con-

| Element | $\boldsymbol{M}_{\mathrm{v,e}}$ | $\boldsymbol{M}_{\mathrm{i,e}}$ | $\boldsymbol{M}_{\mathrm{x,e}}$ | $\boldsymbol{M}_{\dot{\mathrm{x}},\mathrm{e}}$ | $\boldsymbol{M}_{\mathrm{q,e}}$ | $\boldsymbol{u}_{\mathrm{e}}$ | $\boldsymbol{f}_{\mathrm{e}}(\boldsymbol{q}_{\mathrm{e}})$ |
|---|---|---|---|---|---|---|---|
| voltage source $v_{\mathrm{s}}$ | $(1)$ | $(0)$ | $()$ | $()$ | $()$ | $(v_{\mathrm{s}})$ | |
| current source $i_{\mathrm{s}}$ | $(0)$ | $(-1)$ | $()$ | $()$ | $()$ | $(i_{\mathrm{s}})$ | |
| resistor $R$ | $(-1)$ | $(R)$ | $()$ | $()$ | $()$ | $(0)$ | |
| capacitor $C$ | $\begin{pmatrix} C \\ 0 \end{pmatrix}$ | $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ | $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ | $\begin{pmatrix} \\ \end{pmatrix}$ | $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ | |
| inductor $L$ | $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ | $\begin{pmatrix} 0 \\ L \end{pmatrix}$ | $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ | $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ | $\begin{pmatrix} \\ \end{pmatrix}$ | $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ | |
| ideal op amp | $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} \\ \end{pmatrix}$ | $\begin{pmatrix} \\ \end{pmatrix}$ | $\begin{pmatrix} \\ \end{pmatrix}$ | $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ | |
| soft-clipping op amp | $\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} \\ \\ \end{pmatrix}$ | $\begin{pmatrix} \\ \\ \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$ | $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ | $\left(V_{\max} \cdot \tanh\left(\frac{A}{V_{\max}} q_{\mathrm{e},1}\right) - q_{\mathrm{e},2}\right)$ |
| diode | $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ | $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} \\ \end{pmatrix}$ | $\begin{pmatrix} \\ \end{pmatrix}$ | $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ | $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ | $\left(I_{\mathrm{s}} \cdot \left(e^{q_{\mathrm{e},1}/V_{\mathrm{t}}} - 1\right) - q_{\mathrm{e},2}\right)$ |

**Table 1**. Coefficient matrices and non-linear functions of commonly used elements.

trolled voltage source, and also a transformer model that takes into account a non-linear relationship between magnetic field strength and magnetic flux density is non-trivial to map to a voltage/current relationship (giving rise to e.g. the gyrator-capacitor model). In both cases, extensions to the nodal DK method are necessary [6].

While the first two limitations may be easily circumvented, the third means that forming any advanced element model may necessitate extensions to the nodal analysis to cope with controlled voltage sources or to derive a mapping of physical quantities to voltages and currents in a suitable way, or even both. Therefore, in this paper, we present an alternative approach that is more flexible than the nodal DK method and hopefully allows for easier integration of almost arbitrary element models. Instead of nodal analysis, the proposed method employs an analysis technique that is somewhat similar to the sparse tableau approach [8]. The equation system thus obtained is then transformed to a non-linear state space model similar to (1).

## 2. DESCRIPTION OF INDIVIDUAL ELEMENTS

In order to describe individual circuit elements, we will develop a framework that allows a very general formulation of the relationships between voltages, currents, internal states, and the state derivatives. While the definition of voltage and current is obvious for two-pin elements, it needs to be clarified for elements with more pins. In general, we will assume an element with $n$ pins to be internally represented with (at most) $n-1$ branches and consider the voltages across and currents through these branches. These branches may be chosen systematically by choosing one reference pin and adding branches from the reference pin to every other pin (e.g. choose the base pin of a transistor as reference to obtain base-collector and base-emitter branches), or in a more element-specific way (e.g. a primary-side and a secondary side branch for a transformer) as seen fit.

To facilitate precomputation as much as possible, we try to separate linear and non-linear equations and for that purpose introduce auxiliary variables. The non-linear equations are then allowed to depend on these auxiliary variables only. Thus, we can write the mathematical model of any circuit element in the form

$$\boldsymbol{M}_{\mathrm{v,e}}\boldsymbol{v}_{\mathrm{e}} + \boldsymbol{M}_{\mathrm{i,e}}\boldsymbol{i}_{\mathrm{e}} + \boldsymbol{M}_{\mathrm{x,e}}\boldsymbol{x}_{\mathrm{e}} + \boldsymbol{M}_{\dot{\mathrm{x}},\mathrm{e}}\dot{\boldsymbol{x}}_{\mathrm{e}} + \boldsymbol{M}_{\mathrm{q,e}}\boldsymbol{q}_{\mathrm{e}} = \boldsymbol{u}_{\mathrm{e}} \quad (2a)$$

$$\boldsymbol{f}_{\mathrm{e}}(\boldsymbol{q}_{\mathrm{e}}) = \boldsymbol{0}, \quad (2b)$$

where $\boldsymbol{v}_{\mathrm{e}}$ and $\boldsymbol{i}_{\mathrm{e}}$ are vectors of the element's branch voltages and currents, $\boldsymbol{x}_{\mathrm{e}}$ are the element's internal states, $\dot{\boldsymbol{x}}_{\mathrm{e}}$ their derivatives, and $\boldsymbol{q}_{\mathrm{e}}$ the auxiliary variables. The coefficient matrices $\boldsymbol{M}_{\cdot,\mathrm{e}}$, the source vector $\boldsymbol{u}_{\mathrm{e}}$, and the non-linear function $\boldsymbol{f}_{\mathrm{e}}$ then characterize the element. With the number of branches $n_{\mathrm{b,e}}$, the number of states $n_{\mathrm{x,e}}$, and the number of entries $n_{\mathrm{q,e}}$ in $\boldsymbol{q}_{\mathrm{e}}$, the total number of equations, i.e. the number of rows of the matrices plus the number of entries in $\boldsymbol{f}_{\mathrm{e}}$, has be to equal to $n_{\mathrm{b,e}} + n_{\mathrm{x,e}} + n_{\mathrm{q,e}}$.

The respective matrices and functions for some commonly used elements are given in Table 1. The capacitor, for example, is defined by letting the state be charge, i.e. $x_{\mathrm{e}} = C \cdot v_{\mathrm{e}}$ (first row) and $i_{\mathrm{e}} = \dot{x}_{\mathrm{e}}$ (second row). For the op amp models, the first internal branch is the input side, i.e. between non-inverting and inverting input, while the second branch is between the output and a reference node, typically ground. For the soft-clipping op amp model (adapted from [6]), the first row forces the input current to be zero (infinite input impedance), while the second and third row define $q_{\mathrm{e},1}$ and $q_{\mathrm{e},2}$ as input and output voltage, respectively. The non-linear function then models the soft-clipping behaviour with a maximal output swing of $\pm V_{\max}$ and a small-signal gain of $A$. The output current is arbitrary (zero output impedance). It is worth noting that while [6] puts some effort in extending the nodal DK method to incorporate the soft-clipping op amp, it is straight forward in the proposed approach.

## 3. DESCRIPTION OF CIRCUITS

To describe a whole circuit, we first collect the individual element equations. This is achieved by stacking the voltage vectors $\boldsymbol{v}_{\mathrm{e},1}, \boldsymbol{v}_{\mathrm{e},2}, \dots, \boldsymbol{v}_{\mathrm{e},N}$ of the $N$ elements to obtain an overall voltage vector $\boldsymbol{v} = \begin{pmatrix}\boldsymbol{v}_{\mathrm{e},1}^T & \boldsymbol{v}_{\mathrm{e},2}^T & \cdots & \boldsymbol{v}_{\mathrm{e},N}^T\end{pmatrix}^T$, and likewise for the currents $\boldsymbol{i}$, the states $\boldsymbol{x}$, the state derivatives $\dot{\boldsymbol{x}}$, the auxiliary variables $\boldsymbol{q}$, and the source values $\boldsymbol{u}$. Similarly, the coefficient matrices are collected in block diagonal matrices

$$\boldsymbol{M}_{\mathrm{v}} = \begin{pmatrix} \boldsymbol{M}_{\mathrm{v,e},1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{M}_{\mathrm{v,e},2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{M}_{\mathrm{v,e},N} \end{pmatrix} \tag{3}$$

and likewise for $\boldsymbol{M}_{\mathrm{i}}$, $\boldsymbol{M}_{\mathrm{x}}$, $\boldsymbol{M}_{\dot{\mathrm{x}}}$, and $\boldsymbol{M}_{\mathrm{q}}$. Finally, the non-linear functions are gathered as

$$\boldsymbol{f}\begin{pmatrix} \boldsymbol{q}_{\mathrm{e},1} \\ \boldsymbol{q}_{\mathrm{e},2} \\ \vdots \\ \boldsymbol{q}_{\mathrm{e},N} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_{\mathrm{e},1}(\boldsymbol{q}_{\mathrm{e},1}) \\ \boldsymbol{f}_{\mathrm{e},2}(\boldsymbol{q}_{\mathrm{e},2}) \\ \vdots \\ \boldsymbol{f}_{\mathrm{e},N}(\boldsymbol{q}_{\mathrm{e},N}) \end{pmatrix}. \tag{4}$$

Thus, overall, the constraints imposed by the circuit's elements on the circuit's quantities may be expressed as

$$\boldsymbol{M}_{\mathrm{v}}\boldsymbol{v} + \boldsymbol{M}_{\mathrm{i}}\boldsymbol{i} + \boldsymbol{M}_{\mathrm{x}}\boldsymbol{x} + \boldsymbol{M}_{\dot{\mathrm{x}}}\dot{\boldsymbol{x}} + \boldsymbol{M}_{\mathrm{q}}\boldsymbol{q} = \boldsymbol{u} \tag{5a}$$
$$\boldsymbol{f}(\boldsymbol{q}) = \mathbf{0}. \tag{5b}$$

The circuit topology is incorporated by using the Kirchhoff voltage and current laws that can be written as

$$\boldsymbol{T}_{\mathrm{v}}\boldsymbol{v} = \mathbf{0} \tag{6}$$

and

$$\boldsymbol{T}_{\mathrm{i}}\boldsymbol{i} = \mathbf{0}, \tag{7}$$

where $\boldsymbol{T}_{\mathrm{v}}$ and $\boldsymbol{T}_{\mathrm{i}}$ are matrices for independent loop and node (or cut-set) equations, respectively, and can be obtained by well-known methods (e.g. [9]). It should be noted that while the number of rows in $\boldsymbol{T}_{\mathrm{v}}$ and $\boldsymbol{T}_{\mathrm{i}}$ depend on the topology, their sum is always equal to the total number $N_{\mathrm{b}} = \sum_{i=1}^{N} n_{\mathrm{b,e,i}}$ of branches in the circuit.

In the commonly used circuit analysis methods, the Kirchhoff laws are applied after each other. In e.g. nodal analysis, the Kirchhoff voltage law is applied first to introduce node potentials, and the current law is then applied to construct the final equation system. In contrast, we use both Kirchhoff laws simultaneously and write our final equation system as

$$\begin{pmatrix} \boldsymbol{M}_{\mathrm{v}} & \boldsymbol{M}_{\mathrm{i}} & \boldsymbol{M}_{\mathrm{x}} & \boldsymbol{M}_{\dot{\mathrm{x}}} & \boldsymbol{M}_{\mathrm{q}} \\ \boldsymbol{T}_{\mathrm{v}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{T}_{\mathrm{i}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{i} \\ \boldsymbol{x} \\ \dot{\boldsymbol{x}} \\ \boldsymbol{q} \end{pmatrix} = \begin{pmatrix} \boldsymbol{u} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \tag{8a}$$
$$\boldsymbol{f}(\boldsymbol{q}) = \mathbf{0}. \tag{8b}$$

This constitutes an implicit non-linear differential equation system characterizing the circuit's behaviour which could be further manipulated to e.g. derive a continuous-time non-linear state-space model. We will however, apply time-discretization directly to (8).

## 4. DERIVATION OF THE DISCRETE-TIME MODEL

Considering that computing $\dot{\boldsymbol{x}}$ from (8) with all other quantities known may be computationally demanding due to the implicit non-linear equation and that we require solutions on a regular time-grid given by the audio sampling-rate, multi-step methods are more attractive than the otherwise popular Runge-Kutta methods for numerical solution of (8). In the following, we will restrict ourselves to the trapezoidal integration rule, but the same ideas could be adapted for true multi-step methods.

The trapezoidal integration rule is given by

$$\hat{\boldsymbol{x}}(n) = \hat{\boldsymbol{x}}(n-1) + \frac{T}{2}\left(\hat{\dot{\boldsymbol{x}}}(n) + \hat{\dot{\boldsymbol{x}}}(n-1)\right), \tag{9}$$

where $T$ denotes the sampling interval, $\hat{\boldsymbol{x}}(n)$ the approximate state at time $nT$, and $\hat{\dot{\boldsymbol{x}}}(n)$ the exact solution of (8) for $\dot{\boldsymbol{x}}$ using $\boldsymbol{x} = \hat{\boldsymbol{x}}(n)$. We do not, however, apply the integration rule as such, but instead introduce canonical states

$$\bar{\boldsymbol{x}}(n) = \hat{\boldsymbol{x}}(n) + \frac{T}{2}\hat{\dot{\boldsymbol{x}}}(n) \tag{10}$$

which, together with (9), allow to write the substitution rule

$$\hat{\dot{\boldsymbol{x}}}(n) = \frac{1}{T}\left(\bar{\boldsymbol{x}}(n) - \bar{\boldsymbol{x}}(n-1)\right) \tag{11a}$$
$$\hat{\boldsymbol{x}}(n) = \frac{1}{2}\left(\bar{\boldsymbol{x}}(n) + \bar{\boldsymbol{x}}(n-1)\right). \tag{11b}$$

Thus, letting $\boldsymbol{x} = \hat{\boldsymbol{x}}(n)$ and $\dot{\boldsymbol{x}} = \hat{\dot{\boldsymbol{x}}}(n)$ in (8) and introducing

$$\bar{\boldsymbol{M}}_{\mathrm{x}'} = \frac{1}{T}\boldsymbol{M}_{\dot{\mathrm{x}}} + \frac{1}{2}\boldsymbol{M}_{\mathrm{x}} \quad \text{and} \quad \bar{\boldsymbol{M}}_{\mathrm{x}} = \frac{1}{T}\boldsymbol{M}_{\dot{\mathrm{x}}} - \frac{1}{2}\boldsymbol{M}_{\mathrm{x}} \tag{12}$$

leads to the discrete-time system

$$\begin{pmatrix} \boldsymbol{M}_{\mathrm{v}} & \boldsymbol{M}_{\mathrm{i}} & \bar{\boldsymbol{M}}_{\mathrm{x}'} & \boldsymbol{M}_{\mathrm{q}} \\ \boldsymbol{T}_{\mathrm{v}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{T}_{\mathrm{i}} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \bar{\boldsymbol{v}}(n) \\ \bar{\boldsymbol{i}}(n) \\ \bar{\boldsymbol{x}}(n) \\ \bar{\boldsymbol{q}}(n) \end{pmatrix} = \begin{pmatrix} \bar{\boldsymbol{M}}_{\mathrm{x}}\bar{\boldsymbol{x}}(n-1) \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \bar{\boldsymbol{u}}(n) \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \tag{13a}$$

$$f(\bar{q}(n)) = \mathbf{0}, \tag{13b}$$

where $\bar{v}(n)$ shall denote the voltages $v$ at time $nT$ and likewise for $\bar{i}(n)$, $\bar{q}(n)$, and $\bar{u}(n)$.

The total number of equations $2N_\mathrm{b} + N_\mathrm{x} + N_\mathrm{q}$ in (13), where $N_\mathrm{x} = \sum_{i=1}^{N} n_{\mathrm{x,e,i}}$ is the total number of states and $N_\mathrm{q} = \sum_{i=1}^{N} n_{\mathrm{q,e,i}}$ the number of entries in $q$, equals the number of unknowns in $\bar{v}(n)$, $\bar{i}(n)$, $\bar{x}(n)$, and $\bar{q}(n)$, so that (13a) does not have a unique solution on its own (unless the circuit has no non-linear elements and $f$ is empty). We may, however, obtain the general solution of (13a) as

$$\begin{pmatrix} \bar{v}(n) \\ \bar{i}(n) \\ \bar{x}(n) \\ \bar{q}(n) \end{pmatrix} = \begin{pmatrix} D_\mathrm{v} \\ D_\mathrm{i} \\ A \\ D_\mathrm{q} \end{pmatrix} \bar{x}(n-1) + \begin{pmatrix} E_\mathrm{v} \\ E_\mathrm{i} \\ B \\ E_\mathrm{q} \end{pmatrix} \bar{u}(n) + \begin{pmatrix} F_\mathrm{v} \\ F_\mathrm{i} \\ C \\ F_\mathrm{q} \end{pmatrix} z(n) \tag{14}$$

where $z(n)$ is arbitrary with as many elements as $f$. Extracting only the quantities of interest from $\bar{v}(n)$ and $\bar{i}(n)$ to get the output $y(n)$, we thus finally obtain the sought-after state-space system

$$\bar{x}(n) = A\bar{x}(n-1) + B\bar{u}(n) + Cz(n) \tag{15a}$$

$$y(n) = D\bar{x}(n-1) + E\bar{u}(n) + Fz(n) \tag{15b}$$

$$\bar{q}(n) = D_\mathrm{q}\bar{x}(n-1) + E_\mathrm{q}\bar{u}(n) + F_\mathrm{q}z(n) \tag{15c}$$

$$f\big(q(n)\big) = \mathbf{0}, \tag{15d}$$

where $\bar{q}(n)$ and $z(n)$ take a similar role as $i(n)$ and $v(n)$ in (1), but are not equivalent, as (15d) is implicit in contrast to (1d). The computation scheme for execution is nevertheless almost equal: first a suitable $z(n)$ needs to be found such that the result of (15c) is consistent with (15d), which may then be used in (15a) and (15b).

It should be noted that the need to solve an equation in every time-step originates directly from the circuit description in (8) and has nothing to do with the fact that an implicit discretization scheme was used. Using an explicit scheme like forward Euler would not have any advantage.

## 5. COMPARISON TO THE NODAL-DK METHOD

Comparing the proposed approach to [1], the most apparent difference is that the linear system (13a) to solve is much larger than the one in [1] and a general solution for an under-determined system is needed instead of a unique solution. However, this is only one-time effort, and furthermore, the sparsity of the matrix in (13a) can be exploited by schemes like the one of [10].

A more subtle difference is that while $z(n)$ in general has the same number of entries as $i(n)$ in (1) according to [1], $\bar{q}(n)$ in general has more entries than $v(n)$. Thus, the complexity of solving the non-linear equation is slightly higher than in [1] in that computing $\bar{q}(n)$ from $z(n)$ involves more operations than computing $v(n)$ from $i(n)$.

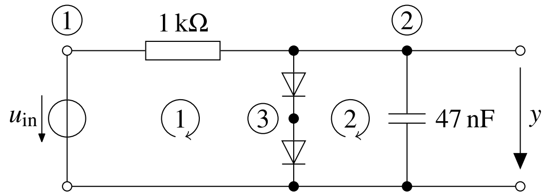


**Fig. 1**. Circuit schematics of a simple diode clipper used as example with nodes and loops numbered.

When considering pre-computing solutions and storing them in a look-up table, it comes at a disadvantage that $p(n) = D_\mathrm{q}\bar{x}(n-1) + E_\mathrm{q}\bar{u}(n)$ of the proposed approach has more dimensions than the corresponding vector of [1], making the required look-up table more complex as it needs more dimensions. However, even for the proposed approach, $p(n)$ will be contained in a lower-dimensional subspace as spanned by $D_\mathrm{q}$ and $E_\mathrm{q}$. Choosing $D_\mathrm{q}$ and $E_\mathrm{q}$ such that they are orthogonal to $F_\mathrm{q}$ in fact allows for look-up tables of the same size as in [1].

On the plus side, the proposed method imposes no restrictions on the circuit to analyze (except that it must be physically plausible) and introducing new circuit element models only means extending Table 1 while no changes to the method itself are required. This generality and flexibility in our opinion outweighs the above-mentioned minor disadvantages.

## 6. EXAMPLE

The proposed method shall be exemplified with the circuit of Figure 1. Note that the nodal DK method could not be applied to the circuit as is due to node three being connected to non-linear elements only.

Ordering the elements as resistor, voltage source, capacitor, left diode, right diode, and referring to Table 1, one can immediately find[1]

$$M_\mathrm{v} = \left(\begin{array}{c|c|c|c|c} -1 & & & & \\ \hline & 1 & & & \\ \hline & & 47\cdot 10^{-9} & & \\ & & 0 & & \\ \hline & & & 1 & \\ & & & 0 & \\ \hline & & & & 1 \\ & & & & 0 \end{array}\right) \quad M_\mathrm{i} = \left(\begin{array}{c|c|c|c|c} 1000 & & & & \\ \hline & 0 & & & \\ \hline & & 0 & & \\ & & 1 & & \\ \hline & & & 0 & \\ & & & 1 & \\ \hline & & & & 0 \\ & & & & 1 \end{array}\right) \quad M_\mathrm{x} = \left(\begin{array}{c} \\ \hline \\ \hline -1 \\ 0 \\ \hline \\ \\ \hline \\ \\ \end{array}\right)$$

$$M_{\dot{\mathrm{x}}} = \left(\begin{array}{c} \\ \hline \\ \hline 0 \\ -1 \\ \hline \\ \\ \hline \\ \\ \end{array}\right) \quad M_\mathrm{q} = \left(\begin{array}{cc|cc} & & & \\ \hline & & & \\ \hline & & & \\ & & & \\ \hline -1 & 0 & & \\ 0 & -1 & & \\ \hline & & -1 & 0 \\ & & 0 & -1 \end{array}\right) \quad u = \left(\begin{array}{c} 0 \\ \hline u_\mathrm{in} \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right), \tag{16}$$

where the all-zero off-diagonal blocks have been omitted for clarity, and

$$f(q) = \begin{pmatrix} I_{\mathrm{s},1}\cdot\left(e^{q_1/V_{\mathrm{t},1}} - 1\right) - q_2 \\ I_{\mathrm{s},2}\cdot\left(e^{q_3/V_{\mathrm{t},2}} - 1\right) - q_4 \end{pmatrix}, \tag{17}$$

[1]All units have been dropped for readability's sake; where applicable, quantities are assumed to be in standard SI units.

where the two diodes may have different reverse saturation currents $I_{\mathrm{s},1}$ and $I_{\mathrm{s},2}$ and threshold voltages $V_{\mathrm{t},1}$ and $V_{\mathrm{t},2}$. The circuit topology with the nodes and loops as indicated in Figure 1 leads to

$$\boldsymbol{T}_{\mathrm{v}} = \begin{pmatrix} 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix} \quad \boldsymbol{T}_{\mathrm{i}} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}. \tag{18}$$

We now apply time-discretization according to (12) for a sampling rate of 44.1 kHz, i.e. with $T = 1/44100$, to obtain

$$\bar{\boldsymbol{M}}_{\mathrm{x}'} = \left(\begin{array}{c} \\ \hline \\ \hline -0.5 \\ -44\,100 \\ \hline \\ \\ \hline \\ \end{array}\right) \qquad \bar{\boldsymbol{M}}_{\mathrm{x}} = \left(\begin{array}{c} \\ \hline \\ \hline 0.5 \\ -44\,100 \\ \hline \\ \\ \hline \\ \end{array}\right). \tag{19}$$

Solving the system of (13a) to obtain a description in the form of (14) and extracting the third entry of $\bar{\boldsymbol{v}}(n)$, the voltage across the capacitor, as output, i.e. using the third row of $\boldsymbol{D}_{\mathrm{v}}$, $\boldsymbol{E}_{\mathrm{v}}$, and $\boldsymbol{F}_{\mathrm{v}}$ as $\boldsymbol{D}$, $\boldsymbol{E}$, and $\boldsymbol{F}$, respectively, then leads to the required state-space coefficient matrices

$$\boldsymbol{A} = \begin{pmatrix} -1 \end{pmatrix} \quad \boldsymbol{B} = \begin{pmatrix} 0 \end{pmatrix} \quad \boldsymbol{C} = \begin{pmatrix} 94 \cdot 10^{-9} & 0 \end{pmatrix} \tag{20a}$$

$$\boldsymbol{D} = \begin{pmatrix} 0 \end{pmatrix} \quad \boldsymbol{E} = \begin{pmatrix} 0 \end{pmatrix} \quad \boldsymbol{F} = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{20b}$$

$$\boldsymbol{D}_{\mathrm{q}} = \begin{pmatrix} 0 \\ 88\,200 \\ 0 \\ 88\,200 \end{pmatrix} \quad \boldsymbol{E}_{\mathrm{q}} = \begin{pmatrix} 0 \\ 1 \cdot 10^{-3} \\ 0 \\ 1 \cdot 10^{-3} \end{pmatrix} \tag{20c}$$

$$\boldsymbol{F}_{\mathrm{q}} = \begin{pmatrix} 1 & -1 \\ -5.1454 \cdot 10^{-3} & 0 \\ 0 & 1 \\ -5.1454 \cdot 10^{-3} & 0 \end{pmatrix}. \tag{20d}$$

These matrices, together with the non-linear function of (17) then define a non-linear state-space system in the form of (15).

For this example, it is obvious that although $\boldsymbol{q}(n)$ has four entries, when constructing a lookup-table, it is sufficient to use the one-dimensional index $p_2 = p_4 = 88\,200 \cdot \bar{\boldsymbol{x}}(n-1) + 1 \cdot 10^{-3} \cdot \bar{u}_{\mathrm{in}}(n)$.

## 7. CONCLUSION

The method presented in this paper allows for a systematic derivation of a non-linear state-space model from circuit schematics. While the nodal analysis-based nodal DK method of [1] achieves the same, it has some limitations and drawbacks. In particular, certain circuit topologies need work-arounds for nodal analysis to be applicable, and circuit elements which cannot be represented as (controlled) current sources can only be included with extensions to the base method. In contrast, the proposed method has no restrictions on the circuit (except for physical plausibility) and allows circuit elements to be described in an extremely flexible way. In fact, the behaviour of circuit elements may be described by an implicit equation in voltages, currents, internal states, and the state derivatives, while at the same time the separation of linear and non-linear parts can be maintained.

The cost for this flexibility and generality is primarily an increase in the matrix sizes during derivation. The final state-space model is comparable to that of [1]. Only one of the intermediate values to be computed when evaluating the non-linear equation is increased in size. In our opinion, the general applicability outweighs this minor disadvantage and we believe the proposed method to be a useful tool for circuit simulation.

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